Comparing Performance Potentials of Classical and Intuitionistic Fuzzy Systems in Terms of *Sculpting the State Space*

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***Abstract*—This article provides new application-independent perspectives about the performance potential of an intuitionistic (I-) fuzzy system over a (classical) Takagi–Sugeno–Kang (TSK) fuzzy system. It does this by extending sculpting the state-space works from a TSK fuzzy system to an I-fuzzy system. It demonstrates that, for piecewise-linear membership functions (trapezoids and triangles), an I-fuzzy system always has significantly more first-order rule partitions of the state space—the coarse sculpting of the state space—than does a TSK fuzzy system, and that some I-fuzzy systems also have more second-order rule partitions of the state space—the fine sculpting of the state space—than does a TSK fuzzy system. It is the author's conjecture that for piecewise-linear membership functions (trapezoids and triangles): it is the always *significantly* greater coarse (and possibly fine) sculpting of the state space that provides an I-fuzzy system with the potential to outperform a TSK fuzzy system, and that a type-1 I-fuzzy system has the potential to outperform an interval type-2 fuzzy system.**

***Index Terms*—Intuitionistic fuzzy sets (I-FS), intuitionistic fuzzy systems, rule partitions, sculpting the state space, Takagi–Sugeno–Kang (TSK) fuzzy systems.**

I. Introduction

RECENTLY, Mendel [1], [2] explained the performance potential[1] of type-1 (T1), interval type-2 (IT2), and general type-2 (GT2) rule-based fuzzy systems (fuzzy systems, for short) that use a singleton fuzzifier and piecewise-linear membership functions (MFs) (trapezoids and triangles) as a *greater sculpting of the state space*. Mendel *et al.* [3] extended [1] to T1 and IT2 fuzzy systems that use a nonsingleton fuzzifier. This article extends [1] to T1 intuitionistic (I-) fuzzy systems.

In a nutshell (more formal discussions are given in Section II), a T1 intuitionistic fuzzy set (I-FS), introduced by Atanassov [4], is described by two functions, a MF and a nonmembership function (NMF), where the constraint MF + NMF ≤ 1 must always be satisfied. A T1 I-fuzzy system is described by two subsets of rules, one for the MFs and the other for the NMFs. Its output can be obtained in either of two ways [5], [6]: 1) the MF and NMF subsystems are coupled by taking a linear combination of their outputs; or 2) the two subsets of rules are viewed as one larger set of rules, and then their outputs are aggregated in the usual way by means of some defuzzification method.

I-FSs have found usefulness in diverse application domains, e.g., aiding human decision making [7]–[12], medical diagnosis [13]–[15], image segmentation [16]–[18], and time series forecasting [19]–[21]. There are many references for T1 (and even IT2) I-FSs and applications for them. We refer the readers to [22]–[24] for very comprehensive discussions as well as many references about them. Due to page limitations, we do not include any of that material herein.

It was recently shown [5], [6], [29], [30] that an I-fuzzy system can outperform a (classical) T1 fuzzy system[2]. We want to be able to understand and explain why this occurs.

Those readers who are already familiar with a T1 I-fuzzy system may answer this by explaining that a T1 I-fuzzy system has more rules and design parameters than a T1 fuzzy system, and using both the MF and NMF permits more flexibility than only using the MF, and so these are reasons for a T1 I-fuzzy system outperforming a T1 fuzzy system.

The goal of this article is to provide further *understanding* of the performance[3] improvement potential of a T1 I-fuzzy system over a T1 fuzzy system, because it is only if such performance improvement potential exists should one even consider using a T1 I-fuzzy system. This goal is accomplished herein by providing new additional explanations for the improved performance in terms of *sculpting the state space* due to using I-FSs.

Manuscript received April 10, 2019; revised July 1, 2019; accepted July 30, 2019. Date of publication August 7, 2019; date of current version September 1, 2020. *(Corresponding author: Jerry M. Mendel.)*

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This article has supplementary downloadable material available at http://ieeexplore.ieee.org, provided by the authors.

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Digital Object Identifier 10.1109/TFUZZ.2019.2933786

[1] As in [1]–[3], interpretability, as a performance metric, is outside of the scope of this article.

[2] The study in [5] is for classification (it uses classification accuracy as the metric) whereas studies in [6], [29], and [30] are for regression (they use RMSE as the metric).

[3] As used here, "performance" is application dependent, e.g., in forecasting it could be a small RMSE, in control it could be low overshoot, in classification it could be low misclassification, etc.; however, the results in this article [1] "provide a common component to all performance analyses, after which the rest of the performance analyses is application dependent."

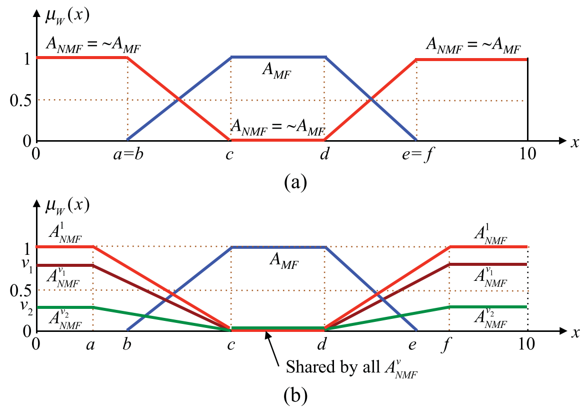

Fig. 1. (a) $A_{\rm MF}/\sim A_{\rm MF}$ and (b) $A_{\rm MF}/A^{\nu}_{\rm NMF}$ in which $A_{\rm MF}$ (blue) is the same for all $A_{\rm MF}/A^{\nu}_{\rm NMF}$ pairs.

The author's conjecture is that for piecewise-linear MFs (trapezoids and triangles): It is the always *significantly* greater coarse (and possibly finer) sculpting of the state space accomplished by using I-FSs that provides a T1 I-fuzzy system with the potential to outperform a T1 fuzzy system.

II. Background

A. I-FS (and Notations)

An *I-FS* A^*, proposed by Atanassov [4], is a generalization of a FS, A; it consists of a MF $\mu_{A^*}(x) \in [0,1]$ and an NMF $\nu_{A^*}(x) \in [0,1]$, where $0 \le \mu_{A^*}(x) + \nu_{A^*}(x) \le 1$, i.e.,

$$A^* = \{(x, \mu_{A^*}(x), \nu_{A^*}(x)) | x \in X,\ 0 \le \mu_{A^*}(x) + \nu_{A^*}(x) \le 1\}. \quad (1)$$

Atanassov [4] also defined the *hesitancy*, $\pi_{A^*}(x)$, of whether x is in A^* or not, as

$$\pi_{A^*}(x) \equiv 1 - [\mu_{A^*}(x) + \nu_{A^*}(x)]. \quad (2)$$

When $\pi_{A^*}(x) = 0$ and $\nu_{A^*}(x) = 1 - \mu_{A^*}(x)$, $A^* \to A$.

When one creates A^* by starting with A, as we shall do, the star notation in (1) can be simplified, i.e., in this article, we use the following notations interchangeably:

$$\begin{cases} \mu_{A^*}(x) \leftrightarrow \mu_A(x) \equiv A_{\rm MF}(x) \text{ or } A^{\mu}(x) \\ \nu_{A^*}(x) \leftrightarrow \nu_A(x) \equiv A_{\rm NMF}(x) \text{ or } A^{\nu}(x). \end{cases} \quad (3)$$

We focus only on *piecewise-linear MFs*, namely trapezoids and triangles, because they are very widely used, and state-space sculpting theory [1] has (so far) only been developed for them.

An NMF is said to be *valid* if and only if $0 \le \mu_{A^*}(x) + \nu_{A^*}(x) \le 1$, which can be satisfied in different ways, i.e., there are different ways to define an NMF for a given MF, e.g., [5] and [25]. One approach is to specify $\pi_{A^*}(x)$ and to then compute $\nu_{A^*}(x)$, whereas another approach is to choose $\nu_{A^*}(x)$ so that the above constraint is satisfied, and (if desired) to then examine the resulting $\pi_{A^*}(x)$. We take the latter approach in this article because it leads to NMFs that are also piecewise linear and are easy to define. More specifically, we adopt Mahapatra and Roy's $A_{\rm MF}/A_{\rm NMF}$ pair (the $A_{\rm MF}/A^1_{\rm NMF}$ pair in Fig. 1(b)) [25]

$$A_{\rm MF}(x) = \begin{cases} \frac{x-b}{c-b}, & \text{for } b \le x \le c \\ 1, & \text{for } c \le x \le d \\ \frac{e-x}{e-d}, & \text{for } d \le x \le e \\ 0, & \text{otherwise} \end{cases}$$

$$A^1_{\rm NMF}(x) = \begin{cases} \frac{c-x}{c-a}, & \text{for } a \le x \le c \\ 0, & \text{for } c \le x \le d \\ \frac{x-d}{f-d}, & \text{for } d \le x \le f \\ 1, & \text{otherwise.} \end{cases} \quad (4)$$

Note that the trapezoid MF in (4) reduces to a triangle MF when $c = d \in [b, e]$.

Theorem 1: 1) $A_{\rm NMF} = A^1_{\rm NMF}$ in (4) is a *valid* NMF; and 2) the sloping legs of $A^1_{\rm NMF}$ and $A_{\rm MF}$ intersect below the grade of 0.5.

Proofs of these results do not appear in [25] and are included in Appendix. The second part of Theorem 1 provides a simple way to quickly "see" if $A^1_{\rm NMF}$ is valid. Observe that $A_{\rm MF}$ in (4) is described by four parameters $\{b, c, d, e\}$ and that $A^1_{\rm NMF}$ is described by only two new parameters $\{a, f\}$.

$A^1_{\rm NMF}$ in (4) is normal; however (also, not in [25]), it can be easily generalized to non-normal NMFs $A^{\nu}_{\rm NMF}$, i.e.,

Corollary 1: Let ν be a scale factor, where $0 < \nu \le 1$; $\nu A^1_{\rm NMF} \equiv A^{\nu}_{\rm NMF}$ [which is described by three new parameters, a, f, and ν; see Fig. 1(b)] is a *valid* NMF, where $A^1_{\rm NMF}$ is in (4).

Proof: Because $A^1_{\rm NMF}$ is a valid NMF (see Theorem 1) and $0 < \nu \le 1, 0 < A_{\rm MF} + \nu A^1_{\rm NMF} \le A_{\rm MF} + A^1_{\rm NMF} \le 1$. Consequently, $\nu A^1_{\rm NMF} \equiv A^{\nu}_{\rm NMF}$ is also a valid NMF.

Corollary 2: When $a = b$, $e = f$, and $\nu = 1$, $A^1_{\rm NMF} = \sim A_{\rm MF}$, i.e., *the NMF is the complement of the MF*.

Proof: This is obvious from Fig. 1(b) and (a).

Much of this article focuses on this case because the parameters that define $A_{\rm MF}$ also define $\sim A_{\rm MF}$.

Definition 1: Points at which an MF or an NMF change its slope are called MF or NMF *kinks*. In this article, to keep things relatively simple, it is assumed that such kinks only occur when a (membership) grade is ν or zero.

Observe, from Fig. 1(a) and (b), that: 1) when $A_{\rm NMF} = \sim A_{\rm MF}$, the kinks of $\sim A_{\rm MF}$ at zero (unity) grade are located at x-values where the kinks of $A_{\rm MF}$ at unity (zero) grades are located; and for all $0 < \nu \le 1$, 2) the two kinks of $A^{\nu}_{\rm NMF}$ and $\sim A_{\rm MF}$ at grade zero occur at the same points, $x = c, d$; and 3) the two kinks of $A^{\nu}_{\rm NMF}$ at grade ν occur at the same points, $x = a, f$, and these locations are different from the comparable two kink locations for $\sim A_{\rm MF}$, which occur at $x = b, e$.

B. Rule-Based Intuitionistic Fuzzy Systems

This article focuses for the most part only on type-1 I-fuzzy systems; hence, we omit "type-1" in our further descriptions.

A rule-based *I-fuzzy system* contains four components—*I-rules*, *I-fuzzifier*, *I-inference (engine)*, and *I-defuzzifier*—that are interconnected as shown in Fig. 2. Once the I-rules have been established, the I-fuzzy system can be viewed as a mapping

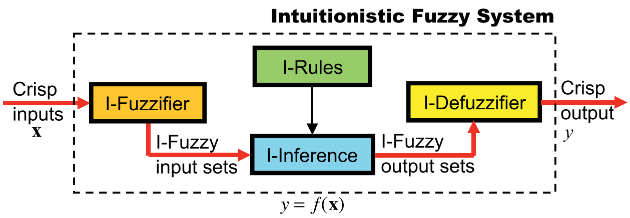

Fig. 2. I-fuzzy system [22].

from inputs to outputs, and this mapping can be expressed quantitatively as $y = f(\mathbf{x})$.

C. I-Rules

An I-fuzzy system uses two subsets of rules, M MF rules and M NMF rules [5], which collectively are its *I-rules* and are stated in (5) for an I-TSK fuzzy system (since sculpting the state space focuses only on the antecedents, the results in the rest of this article are also valid for an I-Mamdani fuzzy system for which $y_k^\mu = b_k^\mu$ and $y_k^\nu = b_k^\nu$). In (5), $k = 1, \ldots, M$, $x_i \in X_i \in \mathbb{R} (i = 1, \ldots, p)$, and the MFs (NMFs) in R_k^μ (R_k^ν) are all given by (4) in which x is replaced by x_i. A TSK fuzzy system only uses the R_k^μ rules

$$\begin{cases} R_k^\mu: & \text{IF} x_1 \text{ is } A_{1k}^\mu \text{ and } \cdots \text{ and } x_p \text{ is } A_{pk}^\mu, \\ & \text{THEN } y_k^\mu = \sum_{i=1}^p w_{ik}^\mu x_i + b_k^\mu \\ R_k^\nu: & \text{IF } x_1 \text{ is } A_{1k}^\nu \text{ and } \cdots \text{ and } x_p \text{ is } A_{pk}^\nu, \\ & \text{THEN } y_k^\nu = \sum_{i=1}^p w_{ik}^\nu x_i + b_k^\nu. \end{cases} \tag{5}$$

D. I-Firing Levels in I-Fuzzy Systems

For an I-fuzzy system, T1 fuzzy logic principles are used to map I-fuzzy input sets in $X_1 \times \cdots \times X_p$ that flow through a set of I-rules into a crisp output, y. We assume *singleton fuzzification*, although the approach that is taken herein is conceptually the same regardless of the nature of the fuzzifier.

It is well known that, for T1 singleton fuzzification, when $\mathbf{x} = \mathbf{x}'$ the *firing levels* for each subset of I-rules are [26], [27] $(k = 1, \ldots, M)$

$$\begin{cases} \text{MF firing levels}: & f_k^\mu(\mathbf{x}') = T_{i=1}^p A_{ik}^\mu(x_i') \\ \text{NMF firing levels}: & f_k^\nu(\mathbf{x}') = T_{i=1}^p A_{ik}^\nu(x_i'). \end{cases} \tag{6}$$

In (6), T denotes a t-norm, usually the minimum or product. Observe that in (6), $\mathbf{x}'$ is processed twice nonlinearly, whereas for a TSK fuzzy system (R_k^μ rules) $\mathbf{x}'$ is only processed once nonlinearly.

Definition 2: An I-firing level $[f_k^\mu(\mathbf{x}')$ or $f_k^\nu(\mathbf{x}')]$ *contributes* to its output only if it is nonzero. This occurs in $X_1 \times \cdots \times X_p$ when the MFs of *all* R_k^μ or R_k^ν antecedents are simultaneously nonzero (i.e., *active* and *firing*).

E. I-Defuzzification

In an I-fuzzy system, after the I-firing levels have been computed, there can be different ways to use them to obtain its final output by means of I-defuzzification [5], [28]. One approach obtains $y(\mathbf{x}')$ by aggregating the M MF and M NMF firing levels simultaneously, as one would do in a TSK fuzzy system. Another approach obtains $y^\mu(\mathbf{x}')$ for the MF subsystem and $y^\nu(\mathbf{x}')$ for the NMF subsystem, and then computes $y(\mathbf{x}') = (1 - \beta)y^\mu(\mathbf{x}') + \beta y^\nu(\mathbf{x}')$. The results that are presented below apply to both approaches.

F. Rule Partitions for TI and I-Fuzzy Systems

One of the important things learned from [1, Sec. III] is that first- and second-order rule partitions of $X_1 \times \cdots \times X_p$ are completely determined by the respective rule partitions of each $(i = 1, \ldots, p)$ X_i separately, because, when minimum or product t-norms are used, if even one component of a rule's firing level is zero then that rule does not contribute to the output of the fuzzy system [e.g., $\min(\text{any } A_{ik}^\mu = 0, \text{ all other } A_{ik}^\mu) = 0$ and $\text{product}(\text{any } A_{ik}^\mu = 0, \text{ all other } A_{ik}^\mu) = 0$]. This carries over as well from fuzzy systems to I-fuzzy systems [see (6)].

Definition 3[1]*:* In a T1 or I-fuzzy system, the *first-order rule partition* of X_i is a collection of nonoverlapping intervals in X_i, in each of which the *same* number of the *same* rules is fired whose firing levels contribute to the output of that system.

Definition 4[1]*:* In a T1 or I-fuzzy system, a *second-order rule partition line* of X_i occurs where the slope of the MF of an FS that is associated with x_i changes its formula within a T1 first-order rule partition of X_i.

Rule partitions *sculpt the state space* into hyper-rectangles within each of which resides a different nonlinear function. First-order rule partitions provide a *coarse sculpting* whereas second-order rule partitions provide a *fine sculpting*.

Many examples of first- and second-order rule partitions for fuzzy systems are in [1] and its supplementary material. For the convenience of the reader, Section I of the supplementary material to this article contains six tables from [1] that provide notations used in first- and second-order rule partitions as well as procedures for establishing them.

III. First-Order Rule Partitions for I-Fuzzy Systems

Because an I-fuzzy system comprises MF and NMF subsystems, we begin by explaining how first-order (MF and NMF) rule partitions are obtained for them, after which we explain how those two kinds of partitions are combined to obtain the first-order rule partitions for the I-fuzzy system.

Assumption A: All results and examples herein are for Q MFs: left and right shoulders and $Q - 2$ interior trapezoidal MFs (not necessarily symmetrical), where each MF intersects only its neighboring left and right MFs once.

Examples in which an MF intersects more than one of its immediate neighboring left or right MFs are given in the supplementary material (Section II, Examples SM-3 and SM-4).

A. First-Order MF Rule Partitions for One Variable

First-order MF rule partition lines occur where an MF first goes from zero to a nonzero value (when an MF becomes "active"), and are located at grade-zero MF kinks.

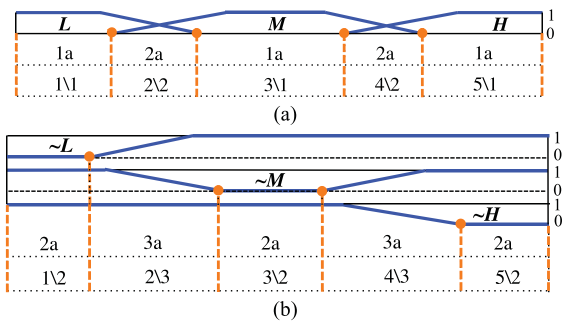

Fig. 3. First-order (a) Example 1 MF and (b) Example 2 NMF rule partitions for one variable, $Q = 3$.

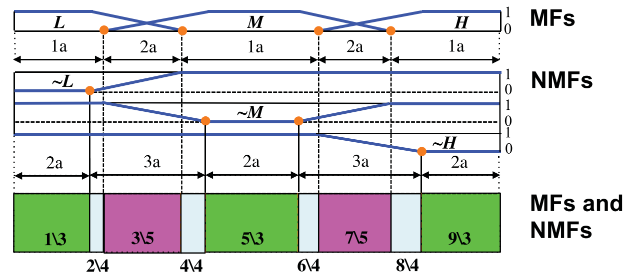

Fig. 4. Example 3 total number of first-order rule partitions in a one-variable I-fuzzy system, $Q = 3$.

Example 1 [1]: Consider $x_i \in [0, 10]$ covered by the three T1 FSs in Fig. 3(a), for which there are three MF rules whose antecedents are R_1^{MF} : IF x_i is L, R_2^{MF} : IF x_i is M, and R_3^{MF} : IF x_i is H. The orange circles in Fig. 3(a) are located at grade-zero MF kinks. The first-order MF rule partition lines are drawn below the orange circles, and extend through two rows. In the first row, in "*m*a" "*m*" is the number of fired rules, and "a" is short for "active." In the second row, in "*l**m*," "*l*" is the numerical name for the first-order MF rule partition, and "*m*" is the number of active rules in that first-order MF rule partition, e.g., 4\2 is partition 4 in which two rules are active. Observe that there are five first-order MF rule partitions, named 1, 2, …, and 5; 1 rule is fired in 1, 2 rules are fired in 2, …, and 1 rule is fired in 5.

Other examples of the first-order MF (and NMF) rule partitions for one variable are given in Section II of the supplementary material.

The partition counts in Fig. 3(a) can be generalized to *Q* MFs.

Theorem 2: Under Assumption A, if the *Q* MFs collectively have K_0 kinks at membership grade zero, then there are $K_0 + 1 = 2Q - 1$ first-order MF rule partitions.

The proof of this theorem appears after the statement of Theorem 3, because the proofs of both theorems are so similar.

Theorem 2 is very useful if one only wants to know the total number of first-order MF rule partitions.

B. First-Order NMF Rule Partitions for One Variable

To construct first-order NMF rule partition lines, it is expedient to first draw each NMF (e.g., $\sim L$, the complement of L) directly above one another, putting the NMF for the left-(right-) most MF at the top (bottom) of the stack. They are not all put on the same plot because they overlap a lot at membership grade unity.

First-order NMF rule partition lines occur where an NMF first goes from zero to a nonzero value (which is when an NMF becomes "active"). These lines are located at grade-zero NMF kinks.

Example 2: This is a continuation of Example 1. The orange circles in Fig. 3(b) are located at grade-zero NMF kinks. The first-order NMF rule partition lines are drawn below the orange circles, and extend through two rows. Observe that in Fig. 3(b) there are five first-order NMF rule partitions, named 1, 2, … and 5, where 2 rules are fired in 1, 3 rules are fired in 2, … and 2 rules are fired in 5. Comparing Fig. 3(a) and (b), observe that more rules are fired in each of the first-order NMF rule partitions than are fired in the first-order MF rule partitions.

The partition counts in Fig. 3(b) can be generalized to *Q* MFs.

Theorem 3: Under Assumption A, if the *Q* MFs collectively have K_1 kinks at membership grade unity, then there are $K_1 + 1 = 2Q - 1$ first-order NMF rule partitions.

Theorem 3 is very useful if one only wants to know the total number of first-order NMF rule partitions.

Proof of Theorems 2 and 3: Shoulder MFs (or NMFs) have one kink at grade zero, whereas interior MFs (or NMFs) have two kinks at grade zero. Consequently, two shoulder and $Q - 2$ interior MFs (or NMFs) have a total of K_0 (or K_1) $= 2Q - 2$ kinks at grade zero, which lead to $K_0 + 1$ (or $K_1 + 1$) $= 2Q - 1$ first-order MF (or NMF) rule partitions (one partition occurs between the origin and the first kink).

Corollary 3: Under Assumption A, if $K_0 = K_1 \equiv K$, then the numbers of first-order MF and NMF rule partitions are the same, namely $K + 1 = 2Q - 1$.

Proof: This follows from Theorems 2 and 3 when $K_0 = K_1 \equiv K$.

Examples 1 and 2 satisfy the conditions of Corollary 2, which is why their numbers of first-order MF and NMF rule partitions are the same.

C. First-Order Rule Partitions for a One Variable I-Fuzzy System

For each x_i, a certain number of MF and NMF rules are fired (active), so that the total number of fired rules in a one-variable I-fuzzy system is the sum of those two numbers, each of which can be read off from MF and NMF first-order rule partition figures.

Example 3: This is a continuation of Examples 1 and 2. Fig. 4 depicts MF, NMF, and combined MF and NMF first-order rule partitions when $Q = 3$. To determine the number or rules fired, stated in the bottom row of Fig. 4, project a vertical line upwards from a partition and add the two "*a*" numbers that it intersects. Observe that the one variable I-fuzzy system has *nine* first-order rule partitions (each with either three, four, or five fired rules in it), which is *almost double* the number of first-order MF (or NMF) rule partitions in an MF (or NMF) fuzzy system (which is 5).

The extensions of Fig. 4 from $Q = 3$ to other values of Q are also given in Section II of the supplementary material.

From all of these examples, one concludes that *there is a lot more coarse-sculpting going on even in a one-variable I-fuzzy system than in a TSK fuzzy system*.

The partition counts in Fig. 4 can be generalized to Q MFs.

Theorem 4: Under Assumption A, in a one-variable I-fuzzy system: 1) there are $4Q - 3$ first-order rule partitions; and 2) if the MFs collectively have K_0 kinks at grade zero and K_1 kinks at grade unity, then there will be $K_0 + K_1 + 1$ first-order rule partitions.

Proof: 1) From the proof of Theorems 2 and 3, two shoulder and $Q - 2$ interior MFs or NMFs collectively have a total of $4Q - 4$ kinks at grade zero. These lead to $4Q - 3$ first-order rule partitions in a one-variable I-fuzzy system (one partition occurs between the origin and the first kink); 2) this follows from the construction procedure for the first-order rule partitions in a one-variable I-fuzzy system, e.g., see Fig. 4. Observe that K_0 dashed lines are drawn downwards from the K_0 MF kinks at MF grade zero, and K_1 dashed lines are drawn downwards from the K_1 NMF kinks at NMF grade zero. Doing this leads to $K_0 + K_1 + 1$ rectangles in the bottom portion of these figures, and those rectangles are the first-order rule partitions in a one-variable I-fuzzy system.

This theorem provides two ways to find the total number of first-order rule partitions in a one-variable I-fuzzy system.

Example 4: Using results from Example 3 and Theorem 4 ($Q = 3$), $4Q - 3 = 9$, which is correct. Fig. 3(a) reveals that $K_0 = K_1 = 4$; hence, $K_0 + K_1 + 1 = 9$, which is also correct. More examples that demonstrate the correctness of $4Q - 3 = K_0 + K_1 + 1$ are in Section III of the supplementary material.

A critic of an I-fuzzy system may argue that of course such a system has more first-order rule partitions than does an MF fuzzy system, because it has twice as many rules. The following theorem demonstrates that it is possible to level the playing field for these two kinds of fuzzy systems, so that they both have exactly the same number of such partitions.

Theorem 5: Under Assumption A, a one-variable TSK fuzzy system that uses Q_{MF} MFs and a one-variable I-fuzzy system that uses Q_I MFs have exactly the same number of first-order rule partitions (their sizes and locations may be different), if

$$Q_{\text{MF}} = 2Q_I - 1. \tag{7}$$

Proof: From Theorems 2 and 4, the numbers of respective first-order rule partitions are $2Q_{\text{MF}} - 1$ and $4Q_I - 3$. Equation (7) is obtained by equating these two numbers and solving for Q_{MF}.

Although it is possible to level the playing field as just described, this comes at a cost to the TSK fuzzy system, in that it will have many more design parameters to tune than does the I-fuzzy system.

Example 5: When $Q_I = 3$, a six-rule I-fuzzy system, whose MFs are the ones depicted in Fig. 3(a), will have (assuming *M* is not symmetrical) eight MF parameters and, if its TSK rule consequents are constants, six consequent parameters, for a total of 14 design parameters, whereas [using (7)] a five-rule TSK fuzzy system, whose five MFs include two shoulders and three interior trapezoids, will have 16 MF parameters, and, if its TSK rule consequents also are constants, five consequent parameters, for a total of 21 design parameters—50% more parameters to tune; for $Q_I = 4$, an eight-rule I-fuzzy system will have a total of 20 design parameters, whereas a seven-rule MF fuzzy system will have a total of 31 design parameters, slightly more than 50% as many design parameters to tune.

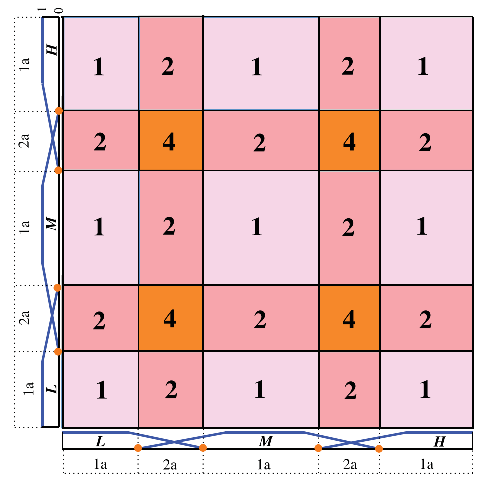

Fig. 5. Example 6 first-order MF rule partitions for two variables, $Q = 3$.

D. First-Order MF Rule Partitions for Two Variables

The construction procedure for creating first-order MF rule partitions for two variables is unchanged from the one that is given in [1, Table III], and is based on constructing the first-order MF rule partitions for X_1 and X_2, and then combining those results [see Section I of the supplementary material (Table SM-III)].

Example 6: This is a continuation of Example 1 [see Fig. 3(a)]. Its results are taken from [1] and are repeated here because they are used in Section III-F. The first-order MF rule partition diagram for $X_1 \times X_2$ is depicted in Fig. 5. The number in each rectangle denotes its number of active MF rules, and is obtained by multiplying the comparable numbers for X_1 and X_2 that are given along the horizontal and vertical axes, respectively. The total number of first-order MF rule partitions is the product of the total number of such partitions (five) for each of the two variables, and is 25.

Theorem 6: Under Assumption A, if X_1 and X_2 are covered by Q_1 and Q_2 MFs, respectively, with $K_0(X_1)$ and $K_0(X_2)$ kinks at grade zero, then there will be $[K_0(X_1)+1]\,[K_0(X_2)+1]$ first-order MF rule partitions of $X_1 \times X_2$.

The proof of this theorem appears after the statement of Theorem 7, because the proofs of both theorems are so similar.

Theorem 6 does not appear in [1] and is very useful when one only wants to know the total number of first-order MF rule partitions of $X_1 \times X_2$

E. First-Order NMF Rule Partitions for Two Variables

The construction procedure for creating first-order NMF rule partitions for two variables also uses the one that is given in [1, Table III], and is based on constructing the first-order

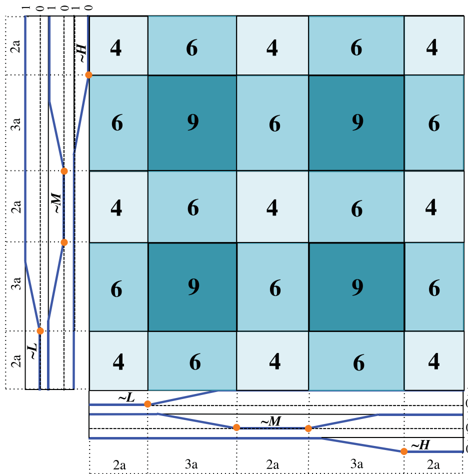

Fig. 6. Example 7 first-order NMF rule partitions for two variables, $Q = 3$.

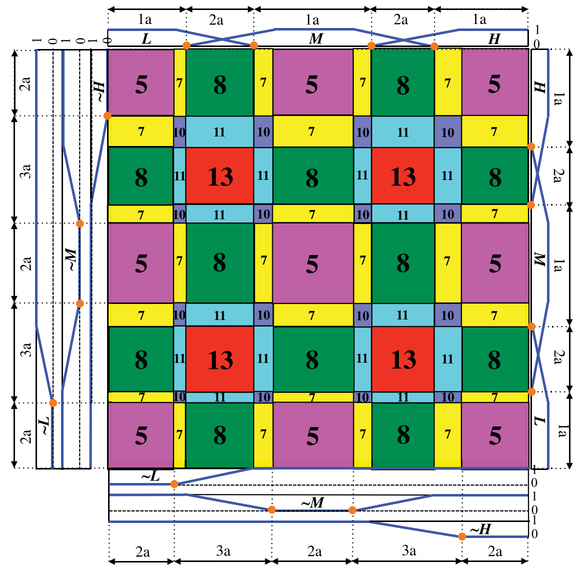

Fig. 7. Example 8 total number of first-order rule partitions in a two-variable I-fuzzy system, $Q = 3$.

NMF rule partitions for X_1 and X_2, and then combining those results.

Example 7: This is a continuation of Example 2 [see Fig. 3(b)]. The first-order NMF rule partition diagram for $X_1 \times X_2$ is depicted in Fig. 6. The number in each rectangle denotes its number of active NMF rules, and is obtained by multiplying the comparable numbers for X_1 and X_2 that are given along the horizontal and vertical axes, respectively. The total number of first-order NMF rule partitions is the product of the total number of such partitions (five) for each of the two variables, and is 25.

Comparing Figs. 5 and 6, observe that, just as for one variable, more rules are fired in each of the first-order NMF rule partitions than are fired in the first-order MF rule partitions.

Theorem 7: Under Assumption A, if X_1 and X_2 are covered by Q_1 and Q_2 MFs, respectively, with $K_1(X_1)$ and $K_1(X_2)$ kinks at grade unity, then there will be $[K_1(X_1)+1][K_1(X_2)+1]$ first-order NMF rule partitions of $X_1 \times X_2$.

Theorem 7 is very useful when one only wants to know the total number of first-order NMF rule partitions on $X_1 \times X_2$.

Proof of Theorems 6 and 7: Theorem 6 (7) follows from Theorem 2 (3) in which K_0 (K_1) is replaced by $K_0(X_1)$ $[K_1(X_1)]$ for X_1 and $K_0(X_2)$ $[K_1(X_2)]$ for X_2, after which [1, eq. (1)] is used. That equation is $N_*^1(X_1, \ldots, X_p) = N_*^1(X_1) \cdots N_*^1(X_p)$, where $N_*^1(X_j)$ is the number of first-order rule partitions of X_j and $*$ is T1 or IT2.

Corollary 4: Under Assumption A, if $K_0 = K_1 \equiv K$, then the numbers of first-order MF and NMF rule partitions on $X_1 \times X_2$ are the same, namely $(K+1)^2 = (2Q-1)^2$.

Proof: This follows from Theorems 6 and 7 when $K_0 = K_1 \equiv K$.

Examples 6 and 7 satisfy the conditions of Corollary 3 ($K = 4$), which is why their numbers of first-order MF and NMF rule partitions are the same.

F. First-Order MF and NMF Rule Partitions for More Than Two Variables

Theorem 8: Under Assumption A and $K_0 = K_1 \equiv K$, let $N_{\text{MF}}^1(X_1, \ldots, X_p)$ and $N_{\text{NMF}}^1(X_1, \ldots, X_p)$, respectively, denote the total number of first-order MF and NMF rule partitions of $X_1 \times \cdots \times X_p$. Then

$$\begin{aligned} N_{\text{MF}}^1(X_1, \ldots, X_p) &= N_{\text{NMF}}^1(X_1, \ldots, X_p) = (2Q-1)^p \\ &= (K+1)^p. \end{aligned} \quad (8)$$

Proof: This is the extension of Corollary 4 from two to p variables.

The extension of (8) to the case when Q is not the same for each X_i is straightforward (use Theorems 6 and 7).

G. First-Order Rule Partitions in a Two-Variable I-Fuzzy System

For each (x_1, x_2) pair, a certain number of MF and NMF rules are fired (active), so that the total number of fired rules in a two-variable I-fuzzy system is the sum of those two numbers, each of which, e.g., for $Q = 3$, is read off from figures like Figs. 5 or 6.

Example 8: This is a continuation of Examples 6 and 7. Fig. 7 depicts MF, NMF, and combined MF and NMF first-order rule partitions when $Q = 3$. Exactly how this figure was obtained from Figs. 5 and 6 is explained in Section III of the supplementary material. Observe that this two variable I-fuzzy system has 81 first-order rule partitions, whereas the "traditional" two-variable TSK fuzzy system only has 25 first-order rule partitions (see Fig. 5), a more than tripling of the number of first-order rule partitions.

Obtaining a figure like the one in Fig. 7 is extremely tedious. The following is a way to bypass this construction if one only wants to know the total number of first-order rule partitions in a two-variable I-fuzzy system.

Theorem 9: Under Assumption A for X_1 and X_2, there are $(4Q-3)^2$ first-order rule partitions in a two-variable I-fuzzy system.

Proof: This follows from Theorem 4 applied to two variables.

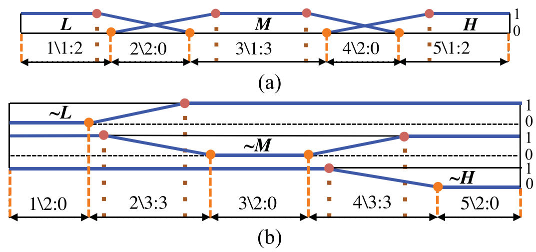

Fig. 8. Second-order (a) Example 10 MFs and (b) Example 11 NMFs rule partitions for one variable, $Q = 3$.

Example 9: When $Q = 3$, $(4Q - 3)^2 = 9^2 = 81$, which agrees with the results in Example 8.

H. Total Number of First-Order Rule Partitions in a p-Variable I-Fuzzy System

Theorem 10: Under Assumption A for each of p variables, when $K_0 = K_1 = K$, the number of first-order rule partitions in a p-variable I-fuzzy system, $N_I^1(X_1, \ldots, X_p)$, is

$$N_I^1(X_1, .., X_p) = (4Q - 3)^p. \tag{9}$$

Proof: This follows from Theorem 4 applied to p variables.

For a TSK fuzzy system, there are [see (8)] $(2Q - 1)^p$ first-order MF rule partitions of $X_1 \times \cdots \times X_p$. Comparing $(2Q - 1)^p$ and $(4Q - 3)^p$, observe that, as Q increases

$$\left(\frac{4Q - 3}{2Q - 1}\right)^p = \left(2 - \frac{1}{2Q - 1}\right)^p \to 2^p. \tag{10}$$

This means that, *as Q increases*, there will be a factor of approximately 2^p more first-order rule partitions of $X_1 \times \cdots \times X_p$ in an I-fuzzy system than the number of such partitions in a TSK fuzzy system, which leads to our conjecture that the hugely greater coarse sculpting of the state space by an I-fuzzy system provides it with the potential to outperform a TSK fuzzy system.

IV. Second-Order Rule Partitions for I-Fuzzy Systems

Second-order rule partitions for an I-fuzzy system can be found from MF and NMF second-order rule partition diagrams. This section is still for $A_{\mathrm{NMF}}^1 = \sim A_{\mathrm{MF}}$.

A. Second-Order MF Rule Partitions for One Variable

Second-order MF rule partition lines occur where the slope of a MF changes within the first-order MF rule partition. These lines occur at grades zero and unity MF kinks, but because MF kinks at grade zero have already contributed the first-order partition lines, new lines are only needed at the unity-grade kinks.

Example 10: This is a continuation of Example 1. Fig. 8(a) begins with Fig. 3(a), after which dotted vertical lines are drawn at grade-unity MF kinks, indicated with brown filled-in circles. The second-order MF rule partitions are drawn below the brown circles and extend into one row. In that row, regarding the notation *l\m:n*, *l* is the numerical name for the first-order MF rule partition; *m* is the number of rules fired in that partition; and *n* are the number of second-order MF rule partitions in it, e.g., 3\1:3 denotes first-order MF rule partition "3," in which "1" rule is fired and in which there are "3" second-order MF rule partitions. Observe that there are 2 second-order MF rule partitions in 1, 0 second-order MF rule partitions in $2, \ldots,$ and 2 second-order MF rule partitions in 5. The total number of second-order MF rule partitions is 7.

B. Second-Order NMF Rule Partitions for One Variable

Second-order NMF rule partition lines occur where the slope of a NMF changes within a first-order NMF rule partition. These lines occur at grades zero and unity NMF kinks, but because grade-zero NMF kinks have already contributed the first-order partition lines, new lines are only needed at the grade-unity kinks.

Example 11: This is a continuation of Example 2. Fig. 8(b) begins with Fig. 3(b) after which dotted vertical lines are drawn at grade-unity kinks, also indicated with brown filled-in circles. The second-order NMF rule partitions are drawn below the brown circles and extend into one row. *The total number of second-order NMF rule partitions is 6.* In this example, the number of second-order NMF rule partitions is less than the number of second-order MF rule partitions. Examples SM-6 and SM-7 in the supplementary material demonstrate that it is also possible for the number of second-order NMF rule partitions to be greater than or equal to the number of second-order MF rule partitions. Such results depend on the relative locations of the grades zero and unity MF and NMF kinks.

C. Second-Order Rule Partitions for a One-Variable I-Fuzzy System

Comparing Fig. 8(a) and (b), observe that first-order MF lines become second-order NMF lines, and second-order MF lines become first-order NMF lines.

Examining the last row of Fig. 4, and making use of the just-stated fact, it should be clear that it is not possible to show the second-order rule partition lines on this figure, because they will lie on top of the first-order MF and NMF lines. This does not mean that second-order rule partitions vanish in the I-fuzzy system. They still exist, but when $A_{\mathrm{NMF}} = \sim A_{\mathrm{MF}}$ they can only be observed on the individual second-order MF and NMF rule partition diagrams, namely on figures like Fig. 8(a) and (b).

D. Second-Order MF Rule Partitions for Two Variables

The construction procedure for creating second-order MF rule partitions for two variables is unchanged from the one that is given in [1, Table VI], and begins with a first-order MF rule partition diagram. It is presented in Table SM-VI in the supplementary material.

Example 12: This is a continuation of Examples 6 and 10 when $Q = 3$. Its results are taken from [1], and are repeated here because they are needed in Section IV-F. The second-order MF rule partition diagram for $X_1 \times X_2$ is depicted in Fig. 9. Note that regarding the notation *m:t*, that appears in each of the

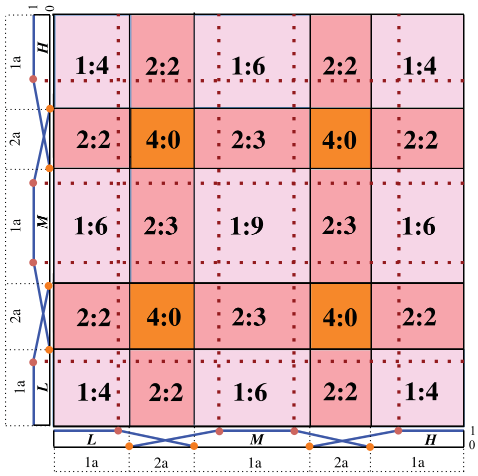

Fig. 9. Example 12 second-order MF rule partitions for two variables, $Q = 3$.

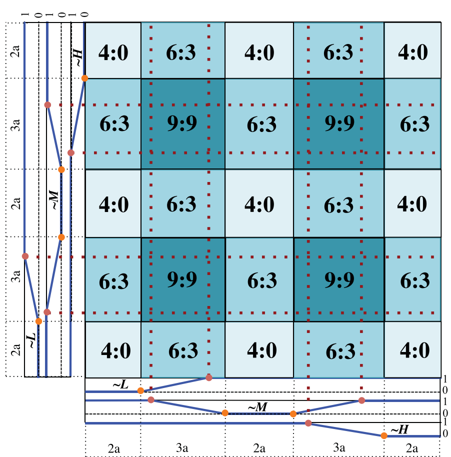

Fig. 10. Example 13 second-order NMF rule partitions for two variables, $Q = 3$.

Fig. 9 rectangles, *m* is the number of rules fired in that first-order MF rule partition; and *t* is the number of second-order MF rule partitions in it, e.g., 2:3 denotes "2" rules are fired and there are "3" second-order MF rule partitions. The total number of second-order MF rule partitions is obtained by counting and is 77.

As is explained in [1, Sec. III-B], there is a formula for computing the total number of second-order MF rule partitions for *p* variables, one that only needs information about the second-order MF rule partitions for *each* variable, and so it is practical. It is given in Section V of the supplementary material.

E. Second-Order NMF Rule Partitions for Two Variables

The construction procedure for creating second-order NMF rule partitions for two variables is also the one that is given in [1, Table VI], and begins with the first-order NMF rule partition diagram.

Example 13: This is a continuation of Examples 7 and 11 when $Q = 3$. The second-order NMF rule partition diagram for $X_1 \times X_2$ is depicted in Fig. 10. The total number of second-order NMF rule partitions is again obtained by counting, and is 72.

A formula for computing the total number of second-order NMF rule partitions for *p* variables is also given in Section V of the supplementary material.

F. Second-Order Rule Partitions for a Two-Variable I-Fuzzy System

Comparing Figs. 9 and 10 observe that once again *first-order MF lines become second-order NMF lines, and second-order MF lines become first-order NMF lines*. Consequently, it is not possible to show the second-order rule partition lines in Fig. 7, because they will lie on top of the first-order MF and NMF lines. This does not mean that second-order rule partitions have disappeared in the I-fuzzy system. They still exist, but, when $(i = 1, 2)$ $A_{\text{NMF}}(x_i) = {\sim}A_{\text{MF}}(x_i)$, they can only be observed on the individual second-order MF and NMF rule partition diagrams, namely on figures like Figs. 9 and 10.

V. Partitions for More General I-Fuzzy Systems

This section extends Sections III and IV results to $A_{\text{MF}}/A^{\nu}_{\text{NMF}}$ in (4), where $A^{\nu}_{\text{NMF}} = \nu A_{\text{NMF}}$ and $0 < \nu \leq 1$ [see Fig. 1(b) and Corollary 1].

A. First-Order Rule Partitions

Theorem 11: All results that are given in Section III about first-order NMF or combined MF and NMF rule partitions are unchanged for all A^{ν}_{NMF}, $0 < \nu \leq 1$.

Proof: First-order MF and NMF rule partitions are found by examining where kinks occur at grade zero, and this theorem is true because they occur at exactly the same points for all A^{ν}_{NMF} [see Fig. 1(b)].

Example 14: Fig. 11 gives first-order partition information for MFs, NMFs, and both MFs and NMFs in a one-variable I-fuzzy system, when MFs and NMFs are given in (4), and ν=1. Comparing Figs. 11 and 4, observe that: 1) the number of first-order NMF and combined MF and NMF rule partitions is unchanged; and 2) it is only the sizes of some of the NMF and combined MF and NMF rule partitions that have changed.

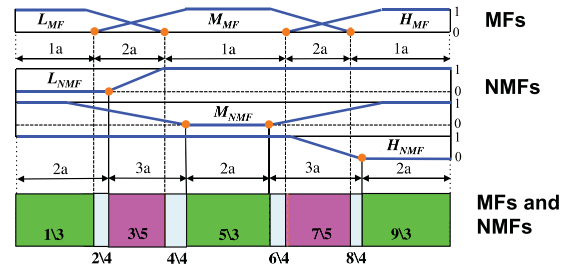

Fig. 11. Example 14 total number of first-order rule partitions in a one-variable I-fuzzy system, when $A^{\nu}_{\text{NMF}} = A^{1}_{\text{NMF}}$ and $Q = 3$.

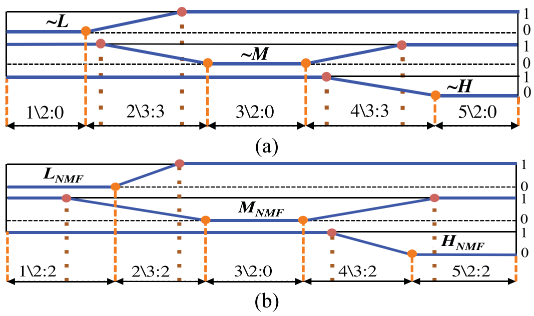

Fig. 12. Example 15 second-order NMF rule partitions for one variable, $Q = 3$: (a) $A_{\text{NMF}} = \sim A_{\text{MF}}$ and (b) $A_{\text{NMF}}^{\nu} = A_{\text{NMF}}^{1}$.

B. Second-Order Rule Partitions

Theorem 12: Second-order rule partitions for A_{NMF}^{ν} $(0 < \nu \leq 1)$ are different from those of $\sim A_{\text{MF}}$.

Proof: Second-order MF and NMF rule partitions are found by examining where kinks occur at grade unity, and they occur at exactly the same points for all A_{NMF}^{ν} [see Fig. 1(b)], but these points are different from those for $\sim A_{\text{MF}}$ [compare Fig. 1(a) and (b)], hence the truth of this theorem.

Example 15: Fig. 12(a) is a repeat of Fig. 8(b), and Fig. 12(b) shows the second-order partition lines for the NMF portion of Fig. 11. Comparing Fig. 12(a) and (b), observe that: 1) there are eight second-order NMF rule partitions in Fig. 12(b) and six in Fig. 12(a); and 2) the locations of some of the second-order NMF rule partitions in Fig. 12(b) are different from the locations in Fig. 12(a). At last, we see a difference between using $\sim A_{\text{MF}}$ and A_{NMF}^{1}, namely, *an I-fuzzy system that uses* A_{NMF}^{1} *can achieve a finer sculpting of the state space than an I- fuzzy system that uses* $\sim A_{MF}$.

Because second-order rule partitions are found by examining where grade-unity kinks occur, and they occur at exactly the same points for all A_{NMF}^{ν}, the results shown in Fig. 12 for the locations and sizes as well as the number of fired rules in each partition are the same for all A_{NMF}^{ν}. It seems, therefore, that the only advantage for using A_{NMF}^{ν}, $\nu \neq 1$, is one additional design degree of freedom, ν.

VI. Observations

Because the numbers of first- and second-order rule partitions only depend on the number of kinks at either zero or unity, or Q (Theorems 2–4, 6–12), partition counts do not depend on (Assumption A) "… where each MF intersects only its neighboring left and right MFs once." They do depend on (Assumption A) "… for Q MFs: left and right shoulders and $Q - 2$ interior trapezoidal MFs (not necessarily symmetrical)." See, e.g., Examples SM-3 and SM-4 in the supplementary material for further verification of this.

Although this article has focused exclusively on T1 fuzzy systems, it is very interesting to compare the number of first- and second-order rule partitions for a T1, IT2, and I-fuzzy system.

TABLE I
Example 16 First- and Second-Order Rule Partitions ($Q = 3$)

Fuzzy system	Number of variables	Number of first-order rule partitions	Number of second-order rule partitions
T1	1	5 (Example 1)	7 (Example 10)
IT2	1	5 [1, Example 1]	9 [1, Example 3]
I-	1	9 (Example 4)	13 (Figs. 8a, b)
T1	2	25 (Example 6)	77 [1, Example 4]
IT2	2	25 [1, Example 6]	135 [1, Example 4]
I-	2	81 (Example 8)	149 (Figs. 9 and 10)

Example 16: We do this for $Q = 3$ in Table I. Observe that for one variable an IT2 (I-) fuzzy system has 5 (9) first-order and 9 (13)[4] second-order rule partitions, and for a two variables an IT2 (I-) fuzzy system has 25 (81) first-order and 135 (149)[5] second-order rule partitions. This (partition theory) suggests that *an I-fuzzy system has the potential to outperform an IT2 fuzzy system*, something that has already been demonstrated in [29] and [30], but for Gaussian T1 and IT2 MFs.

We conjecture (from partition theory) the following ordering for performance improvement for fuzzy systems:

$$\text{T1} < \text{IT2} < \text{T1 Intuitionistic} < \text{GT2} < \text{IT2 Intuitionistic} < \text{GT2 Intuitionistic}. \quad (11)$$

VII. Conclusion

This article has provided new application-independent perspectives about the performance potential of an intuitionistic (I-) fuzzy system over a (classical TSK) fuzzy system, by extending sculpting the state-space works from a fuzzy system to an I-fuzzy system. It has demonstrated that for piecewise-linear MFs (trapezoids and triangles), an I-fuzzy system always has significantly more first-order rule partitions of the state space—the *coarse sculpting* of the state space—than does a TSK fuzzy system, and that some I-fuzzy systems also have more second-order rule partitions of the state space—the *fine sculpting* of the state space—than does a TSK fuzzy system. It is the author's conjecture that: it is the always significantly greater coarse (and possibly finer) sculpting of the state space that provides an I-fuzzy system with the potential to outperform a TSK fuzzy system.

Rule partition theory demonstrates the potential for improved performance, but it is the architecture of an I-fuzzy system that attempts to achieve this. Choosing NMFs as the complements of the MFs introduces no new design parameters; choosing them as in (4) introduces two new design parameters for each NMF; and choosing them by using a scaled version of (4) introduces yet one more design parameter.

Some open research questions and extensions to this article are as follows.

[4]This number was obtained by adding the numbers of second-order rule partitions in Fig. 8(a) and (b).

[5]This number was obtained by adding the numbers of second-order rule partitions in Figs. 9 and 10.

1) *Study how much performance improvement occurs when one uses an NMF other than the complement of the MF.* This may be application dependent.
2) *Extend all results to IT2 I-fuzzy systems.* We do not expect any surprises, i.e., an IT2 fuzzy system usually has the same number of first-order rule partitions as does a T1 fuzzy system (although they are of different sizes), but an IT2 fuzzy system usually has many more second- order rule partitions than does a T1 fuzzy system. Consequently, an IT2 fuzzy system usually has a much greater fine sculpting capability of the state space than does a T1 fuzzy system. We expect comparable results between T1 and IT2 I-fuzzy systems.
3) *Extend all results to GT2 I-fuzzy systems*. Such systems have not appeared yet.
4) *Extend all results to nonsingleton T1 and IT2 I-fuzzy systems.* Such systems also have not appeared yet.
5) *Demonstrate (or not) the performance ordering for fuzzy systems that is given in (*11*).*
6) *Extend all sculpting results to Gaussian- (and bell-) shaped MFs.* This needs to be done not only for I-fuzzy systems but also for all other kinds of fuzzy systems.
7) *Study (suggested by a reviewer) the connections, and provide some new ones, between sculpting the state space and universal approximation for I-fuzzy systems.*
8) *Develop rule-partition mathematics.* Drawing partition diagrams can be tedious. Mathematics for determining the number of first- and second-order rule partitions for any value of Q and $p = 1, 2$ is already under development and will be reported on in another article.
9) *Extend patch learning [31] (PL) to I-fuzzy systems.* A patch is a first-order rule partition. PL consists of three steps: 1) train an initial global model using all training data; 2) identify from the initial global model the patches which contribute the most to the learning error, and train a (local) patch model for each such patch; and 3) update the global model using training data that do not fall into any patch. Wu and Mendel [31] have demonstrated the effectiveness of PL on five regression problems, but to-date only for T1 fuzzy systems.

APPENDIX
PROOF OF THEOREM 1

1) A^1_{NMF} is a valid NMF if

$$A^1_{\text{NMF}}(x) + A_{\text{MF}}(x) \leq 1. \qquad \text{(A-1)}$$

Examining Fig. 1(b), it is clear that (A-1) is satisfied when $x \in [0, b]$, $x \in [c, d]$, and $x \in [e, 10]$, so one only needs to see if (A-1) is satisfied for $x \in [b, c]$ and $x \in [d, e]$. The proof is only provided here for $x \in [b, c]$ and $\nu = 1$, for which one must examine

$$(x - b)/(c - b) + (c - x)/(c - a) \overset{?}{\leq} 1\,. \qquad \text{(A-2)}$$

It is straightforward to show, by means of simple algebra that (A-2) reduces to $x \overset{?}{\leq} c$ which is always true for $x \in [b, c]$; hence, (A-1) is satisfied when $x \in [b, c]$.

2) Focusing on $x \in [b, c]$, one needs to first locate the intersection point of A^1_{NMF} and A_{MF}, say x^*, which is found by solving the equation $(x^* - b)/(c - b) = (c - x^*)/(c - a)$, after which the intersection point, $A_{\text{MF}}(x^*)$, is computed, i.e., $x^* = (c^2 - ab)/(2c - a - b)$ and $A_{\text{MF}}(x^*) = (c - b)/(2c - a - b)$.

To prove that $A_{\text{MF}}(x*) < 0.5$, assume $A_{\text{MF}}(x^*) \geq 0.5$ and show that this leads to a contradiction, i.e.,

$$(c - b)/(2c - a - b) \geq 0.5 \Rightarrow b - a \leq 0. \qquad \text{(A-3)}$$

Examining Fig. 1(b), observe that $b - a > 0$; hence, (A-3) is false, which means that $A_{\text{MF}}(x^*) < 0.5$ is true.

ACKNOWLEDGMENT

The authors would like to thank the reviewers of this article who made very useful suggestions that have led to an improved version of this article.

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Supplementary Materials

for

"Comparing the Performance Potentials of Classical and Intuitionistic Fuzzy Systems in Terms of *Sculpting the State Space*"

by

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Imo Eyoh , *Member, IEEE*
Robert John, *Senior Member, IEEE*

In this document we provide Supplementary Material (SM) that includes: six tables from[1] [1], specialized to type-1 (T1) fuzzy systems, more examples of first-order MF and NMF rule partitions for one variable, explaining how Fig. 7 was obtained, more examples of second-order MF and NMF rule partitions for one variable, and formulas for computing the total number of second-order NMF and NMF rule partitions for p variables. Each section begins on a new page.

I. Six Tables From [1] Specialized to Type-1 Fuzzy Systems

The following tables are included here for the convenience of the readers. They apply equally for (classical TSK) fuzzy systems and I-fuzzy systems. Since the notations herein are not use in the main body of this paper, we leave it to the reader to establish notations that let I-fuzzy system quantities be distinguished from fuzzy system quantities.

After [1] was published it was realized that some of the subscript notations could be greatly simplified without the loss of generality, namely subscript x_i could be simplified to subscript i. We have not made this simplification here so that the reader can more easily connect to [1].

TABLE SM-I
NOTATIONS USED FOR FIRST-ORDER RULE PARTITIONS

	First-Order Rule Partitions
Symbol	Definition $(i = 1,\dots,p)$
$P^1(k_{x_i} \mid x_i)$	T1 first-order rule partition of X_i
k_{x_i}	Counter/index of T1 first-order rule partition of X_i ; $k_{x_i} = 1,\dots,N^1(X_i)$
$N^1(X_i)$	Total number of T1 first-order rule partitions of X_i
$N_R(k_{x_i})$	Fixed number of same rules fired in each $P^1(k_{x_i} \mid x_i)$
$P^1(k_{x_1}, k_{x_2},\dots,k_{x_p})$	T1 first-order rule partition of $X_1 \times X_2 \times \cdots \times X_p$, numbered $(k_{x_1}, k_{x_2},\dots,k_{x_p})$
$N^1(X_1, X_2,\dots,X_p)$	Total number of T1 first-order rule partitions of $X_1 \times X_2 \times \cdots \times X_p$ (use (6) in [1])
$N_R(k_{x_1}, k_{x_2},\dots,k_{x_p})$	Fixed number of rules that are fired in each $P^1(k_{x_1}, k_{x_2},\dots,k_{x_p})$ (use (7) in [1])

[1] Reference numbers refer to the references that are at the end of this SM.

TABLE SM-II

TWO-STEP PROCEDURE FOR ESTABLISHING T1 FIRST-ORDER RULE PARTITION QUANTITIES FOR A SINGLE VARIABLE, x_i, IN A T1 FUZZY SYSTEM, ON A PLOT (SKETCH) OF ITS MFS

Step	Description
1	Scan the axis of x_i with an imaginary dashed vertical line from left to right. Count the number of intersections of this line with the MFs of x_i; they represent the number of *same*-rules $[N_R(k_{x_i})]$ whose firing levels contribute to the output of a T1 fuzzy system. When this number, or the nature of the same rules, changes draw a dashed vertical line; it represents the boundary of a T1 first-order rule partition. Insert a dashed vertical line at the start and at the end of X_i. For each x_i, the interval of real numbers between adjacent dashed vertical lines is its T1 first-order rule partition.
2	Count the number of $P^1(k_{x_i} \mid x_i)$, the total being $N^1(X_i)$; then, $k_{x_i} = 1,...,N^1(X_i)$.

TABLE SM-III

FOUR-STEP PROCEDURE FOR ESTABLISHING T1 FIRST-ORDER RULE PARTITION QUANTITIES FOR $X_1 \times X_2$ IN A T1 FUZZY SYSTEM

Step	Description
1	Locate the T1 first-order rule partitions of x_1 (x_2) on the horizontal (vertical) axis, and establish $N_R(k_{x_1})$, $N_R(k_{x_2})$, $N^1(X_1)$ and $N^1(X_2)$
2	Extend all dashed T1 first-order rule partitions (turning them into solid lines) so that they cover $X_1 \times X_2$. The results from doing this will be a collection of rectangles (or squares).
3	Compute $N_R(k_{x_1}, k_{x_2})$ using (7) in [1].
4	Compute $N^1(X_1, X_2)$ using (6) in [1].

TABLE SM-IV

NOTATION USED FOR SECOND-ORDER RULE PARTITIONS

Second-Order Rule Partitions	
Symbol	Definition $(k_{x_i} = 1,...,N^1(x_i))$
$P^2(k_{x_i}, m_{k_{x_i}} \mid x_i)$	T1 second-order rule partition of X_i, often abbreviated to $(k_{x_i}, m_{k_{x_i}})$
$m_{k_{x_i}}$	Counter/index of T1 second-order rule partition of X_i; $m_{k_{x_i}} = 1,...,N^2(k_{x_i} \mid x_i)$
$N^2(k_{x_i} \mid x_i)$	Total number of T1 second-order rule partitions within a T1 first-order rule partition k_{x_i} of X_i
$N^2(X_i)$	Total number of T1 second-order rule partitions of X_i
$P^2((k_{x_1}, k_{x_2}), m_{(k_{x_1}, k_{x_2})})$	T1 second-order rule partition of $X_1 \times X_2$
$m_{(k_{x_1}, k_{x_2})}$	Counter/index of T1 second-order rule partition of $X_1 \times X_2$; $m_{(k_{x_1}, k_{x_2})} = 1,...,N^2(k_{x_1}, k_{x_2})$
$N^2(k_{x_1}, k_{x_2})$	Total number of T1 second-order rule partitions within the $(k_{x_1}, k_{x_2})^{\text{th}}$ T1 or IT2 first-order rule partition of $X_1 \times X_2$
$Z(X_i)$	Number of times that $N^2(k_{x_i} \mid x_i) = 0$
$N^{2'}(X_i)$	$N^2(X_i) + Z(X_i)$
$N^2(X_1, X_2, \ldots, X_p)$	Total number of T1 second-order rule partitions of $X_1 \times X_2 \times \cdots \times X_p$ (use (13) in [1])

TABLE SM-V

FOUR-STEP PROCEDURE FOR ESTABLISHING T1 SECOND-ORDER RULE PARTITION QUANTITIES FOR A SINGLE VARIABLE x_i, IN A T1 FUZZY SYSTEM, ON A PLOT (SKETCH) OF ITS RESPECTIVE FIRST-ORDER RULE PARTITIONS

Step	Description
1	Scan the axis of x_i with an imaginary dotted vertical line from left to right. Wherever a MF changes its formula, draw a dotted vertical line. If the change in formula occurs at a boundary of a T1 first-order rule partition, then do not draw such a vertical dotted line.
2	The interval of real numbers between adjacent dotted vertical lines or between a dotted line and a dashed (or dashed and dotted) line is its T1 second-order rule partition $[P^2(k_{x_i}, m_{k_{x_i}} \mid x_i)]$.
3	Each T1 first-order rule partition has from zero to a finite number of T1 second-order rule partitions [$N^2(k_{x_i} \mid x_i)$].
4	Count the total number of $N^2(k_{x_i} \mid x_i)$, the total being $N^2(X_i)$.

TABLE SM-VI

FOUR-STEP PROCEDURE FOR ESTABLISHING T1 SECOND-ORDER RULE PARTITION QUANTITIES FOR $X_1 \times X_2$ IN A T1 FUZZY SYSTEM, ON A PLOT (SKETCH) OF THE FIRST-ORDER RULE PARTITIONS.

Step	Description
1	Locate the T1 (IT2) second-order rule partitions of x_1 (x_2) on the horizontal (vertical) axis.
2	Extend all dotted T1 second-order rule partitions so that they cover $X_1 \times X_2$. The results from doing this will be a collection of rectangles (or squares).
3	Each T1 first-order rule partition on $X_1 \times X_2$ has from zero to a finite number of T1 second-order rule partitions. Establish $N^2(k_{x_1}, k_{x_2})$ by counting.
4	Count the total number of $N^2(k_{x_1}, k_{x_2})$, the total being $N^2(X_1, X_2)$.

II. More Examples of First-order MF and NMF Rule Partitions for One Variable

Additional examples of first and second-order rule partitions are given in this section.

Example SM-1 (First-order rule partitions, Q = 4): This example is a combination of Examples 1–3 in the main body of this paper, for $Q = 4$.

Now $x_i \in [0,10]$ is covered by the four T1 FSs in Fig. SM-1a, for which there are four MF rules whose antecedents are R_1^{MF} : IF x_i is L, R_2^{MF} : IF x_i is $M1$, R_3^{MF} : IF x_i is $M2$, and, R_4^{MF} : IF x_i is H. Observe that there are seven first-order MF rule partitions, named 1, 2, … and 7; 1 rule is fired in 1, 2 rules are fired in 2, … and 1 rule is fired in 7.

Observe in Fig. SM-1b there are also seven first-order NMF rule partitions, named 1, 2, … and 7, where 3 rules are fired in 1, 4 rules are fired in 2, … and 3 rules are fired in 7. Comparing Figs. SM-1a and SM-1b, observe that (just as was true for Q = 3) *more rules are fired in each of the first-order NMF rule partitions than are fired in the first-order MF rule partitions*.

Fig. SM-2 depicts MF, NMF and combined MF and NMF first-order rule partitions when $Q = 4$. To determine the number or rules fired, stated in the bottom row of Fig. SM-2, project a vertical line upwards from a partition and add the two "a" numbers that it intersects. Observe that the one variable I-fuzzy system has *13* first-order rule partitions (each with either 4, 5 or 6 fired rules in it), which is *almost double* the number of first-order MF (or NMF) rule partitions in a MF (or NMF) fuzzy system (which is 7).

Finally, note that $4Q-3=13$, $K_0 = K_1 = 6$ and $K_0 + K_1 + 1 = 13$, which again demonstrates the truth of Theorem 4.

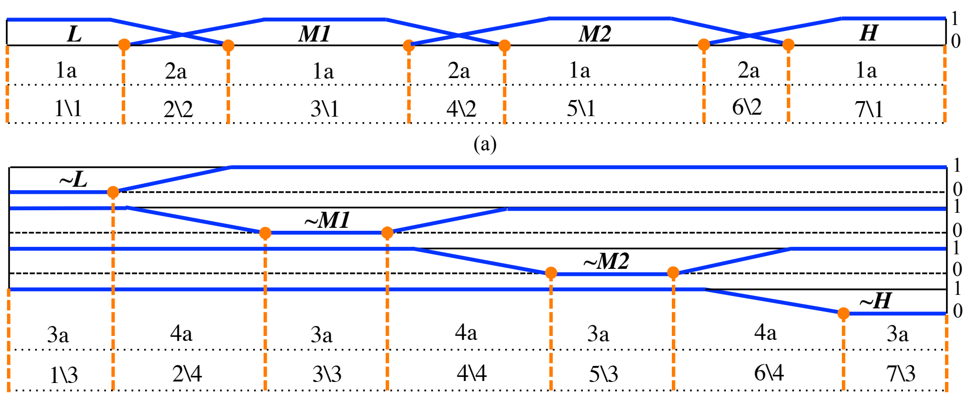


Fig. SM-1. First-order (a) MF and (b) NMF rule partitions for one variable, Q = 4.

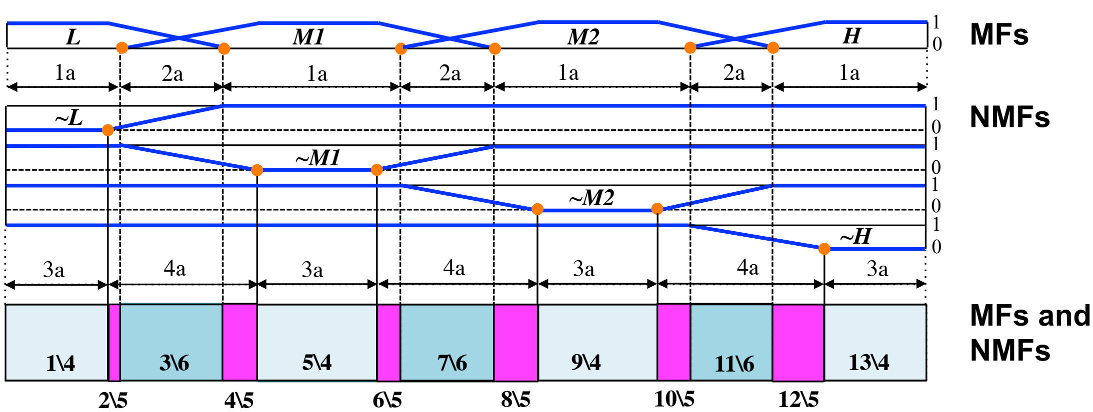


Fig. SM-2. Total number of first-order rule partitions in a one-variable I-fuzzy system, $Q = 4$.

Example SM-2 (First-order rule partitions, Q = 5): This example is analogous to Example SM-1, but for $Q = 5$. Observe in Fig. SM-3, that:

1. There are nine first-order MF rule partitions (names of the partitions are not shown); 1 rule is fired in 1, 2 rules are fired in 2, … and 1 rule is fired in 9.
2. There are also 9 first-order NMF rule partitions, where 4 rules are fired in 1, 5 rules are fired in 2, … and 4 rules are fired in 9.
3. *Again more rules are fired in each of the first-order NMF rule partitions than are fired in the first-order MF rule partitions*.
4. The one variable I-fuzzy system has *17* first-order rule partitions (each with either 5, 6 or 7 fired rules in it), which again is *almost double* the number of first-order MF (or NMF) rule partitions in a MF (or NMF) fuzzy system (which is 9).
5. Finally, note that $4Q-3=17$, $K_0 = K_1 = 8$ and $K_0 + K_1 + 1 = 17$, which again demonstrates the truth of Theorem 4.

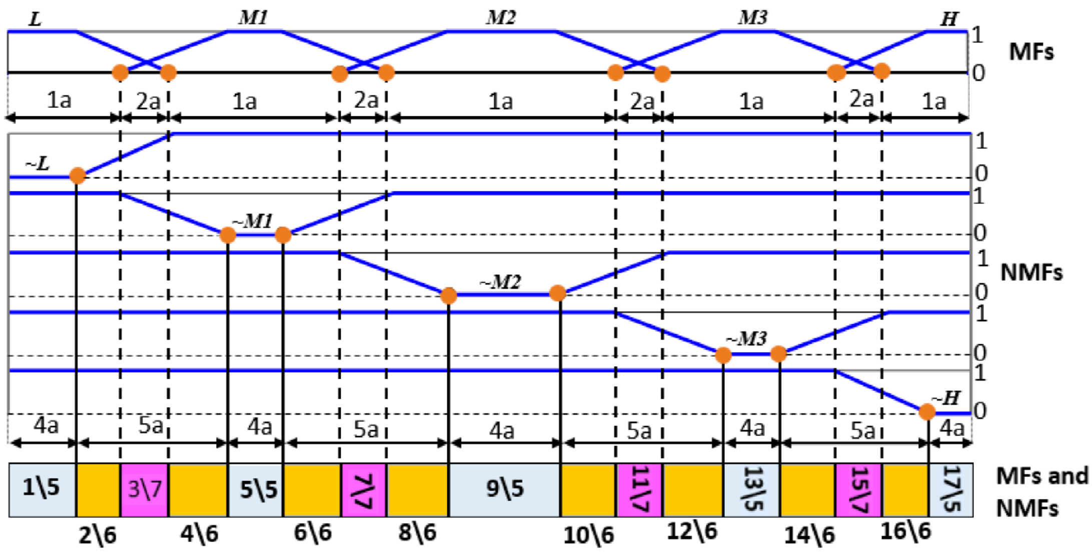


Fig. SM-3. First-order MF and NMF rule partitions for one variable, and total number of first-order rule partitions in a one-variable I-fuzzy system, $Q = 5$.

Example SM-3 (First-order rule partitions, Q = 3 and Assumption A is No Longer True): Recall Assumption A:

> All results and examples herein [in the main body] are for Q MFs: left and right shoulders and $Q-2$ interior trapezoidal MFs (not necessarily symmetrical), where each MF intersects only its neighboring left and right MFs once.

This example breaks the part of this assumption about intersecting MFs, and examines what happens when it is broken.

Observe in Fig. SM-4, that (compare the items below with those for Fig. 4 that are in the main body of this paper):

1. *L* overlaps with *M* and *H*; *M* only overlaps with *L* and *H*; and, *H* overlaps with *L* and *M*.
2. There are still five first-order MF rule partitions (names of the partitions are not shown); 1 rule is fired in 1, 2 rules are fired in 2, *3 rules are fired in 3* (more rules are fired in this partition due to the greater overlap), 2 rules are fired in 4, and 1 rule is fired in 5.
3. There are still five first-order NMF rule partitions, where 2 rules are fired in 1, 3 rules are fired in 2, 2 rules are fired in 3, 3 rules are fired in 4, and 2 rules are fired in 9.
4. *It is no longer true that more rules are fired in each of the first-order NMF rule partitions than are fired in the first-order MF rule partitions*. It is true that *more rules are fired in four of the five first-order NMF rule partitions than are fired in the comparable first-order MF rule partitions*.
5. The one variable I-fuzzy system still has *9* first-order rule partitions (each with either 3, 4, 5 or 6 fired rules in it; this has changed), which again is *almost double* the number of first-order MF (or NMF) rule partitions in a MF (or NMF) fuzzy system (which is 5).
6. Although the number of first-order rule partitions in the one variable I-fuzzy system has not changed (when Assumption A is violated) the number of rules fired in two of the partitions (4 and 5) has changed.

Finally, note that $4Q-3=13$, $K_0=K_1=6$ and $K_0+K_1+1=13$, which again demonstrates the truth of Theorem 4.

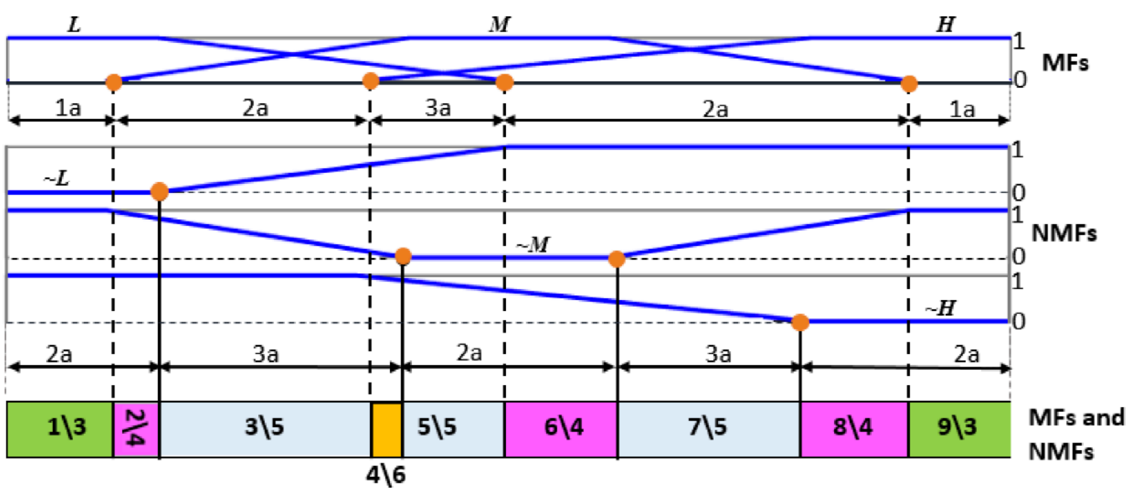


Fig. SM-4. First-order MF and NMF rule partitions for one variable, and total number of first-order rule partitions in a one-variable I-fuzzy system, $Q=3$, when Assumption A is broken.

Example SM-4 (First-order rule partitions, Q = 3 and Assumption A is Again No Longer True): Recall Assumption A:

> All results and examples herein [in the main body] are for Q MFs: left and right shoulders and $Q-2$ interior trapezoidal MFs (not necessarily symmetrical), where each MF intersects only its neighboring left and right MFs once.

This example breaks the part of this assumption about intersecting MFs, more severely than in Example SM-3, and examines what happens when it is broken.

Observe in Fig. SM-5, that:

1. L overlaps with M (twice) and H (once); the left and right legs of M overlap (once) with both L and H; and, H overlaps with L (once) and M (twice).
2. There are still five first-order MF rule partitions (names of the partitions are not shown); 1 rule is fired in 1, 2 rules are fired in 2, *3 rules are fired in 3* (more rules are fired in this partition due to the greater overlap), 2 rules are fired in 4, and 1 rule is fired in 5.
3. There are still five first-order NMF rule partitions, where 2 rules are fired in 1, 3 rules are fired in 2, 2 rules are fired in 3, 3 rules are fired in 4, and 2 rules are fired in 9.
4. *It is no longer true that more rules are fired in <u>each</u> of the first-order NMF rule partitions than are fired in the first-order MF rule partitions*. It is again true that *more rules are fired in four of the five first-order NMF rule partitions than are fired in the comparable first-order MF rule partitions*.
5. The one variable I-fuzzy system still has *9* first-order rule partitions (each with either 3, 4, 5 or 6 fired rules in it; this has changed), which again is *almost double* the number of first-order MF (or NMF) rule partitions in a MF (or NMF) fuzzy system (which is 5).
6. Although the number of first-order rule partitions in the one variable I-fuzzy system has not changed (when Assumption A is violated) the number of rules fired in two of the partitions (4 and 6) has changed.

Finally, note that $4Q-3=13$, $K_0=K_1=6$ and $K_0+K_1+1=13$, which again demonstrates the truth of Theorem 4.

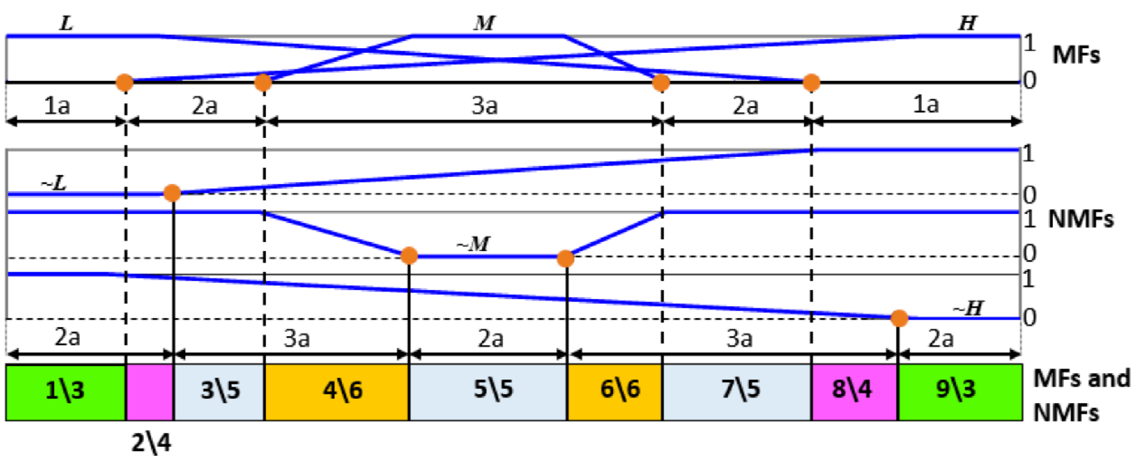


Fig. SM-5. First-order MF and NMF rule partitions for one variable, and total number of first-order rule partitions in a one-variable I-fuzzy system, $Q=3$, when Assumption A is again broken.

III. Explaining How Fig. 7 Was Obtained

Fig. 7 is complicated, and may appear overwhelming at first, because there is so much on it. It is therefore helpful to explain how it was drawn.

1. Begin with Fig. 6, and remove all of its numbers and colorings, but keep all of the horizontal and vertical lines. Use two-way arrows to demarcate the respective first-order NMF rule partitions. This is Fig. SM-6.

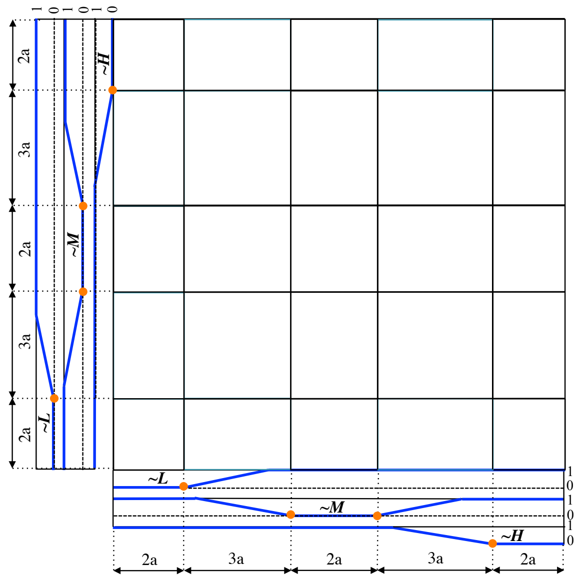


Fig. SM-6. Step 1.

2. Move the horizontal MF labels from the bottom of Fig. 5 to the top of Fig. SM-6 so that the MFs and the orange circles touch the top of the array. Move a flipped version of the vertical MF labels from the left side of Fig. 5 to the right side of Fig. SM-6. This is Fig. SM-7. Use two-way arrows to demarcate the respective first-order rule MF partitions. The Fig. 5 left-side vertical MF labels were flipped so that as one scans the four sides of Fig. SM-7, the MFs appear to be in their proper locations.

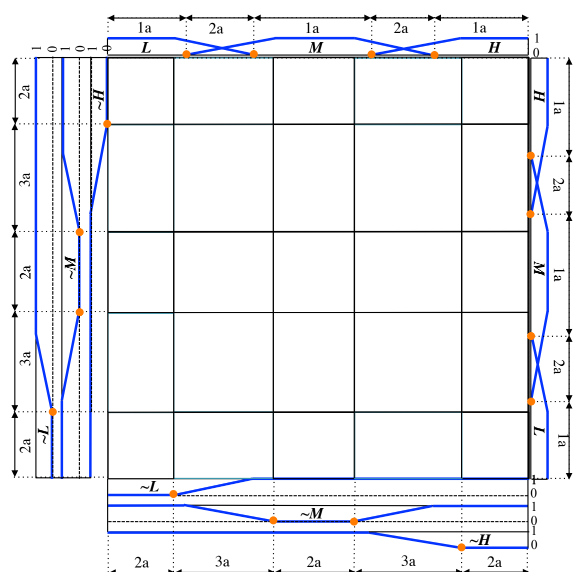


Fig. SM-7. Step 2.

3. Draw horizontal and vertical lines on Fig. SM-7 from all of the orange circles that are on its top and right sides. There are 81 rectangles. This is Fig. SM-8.

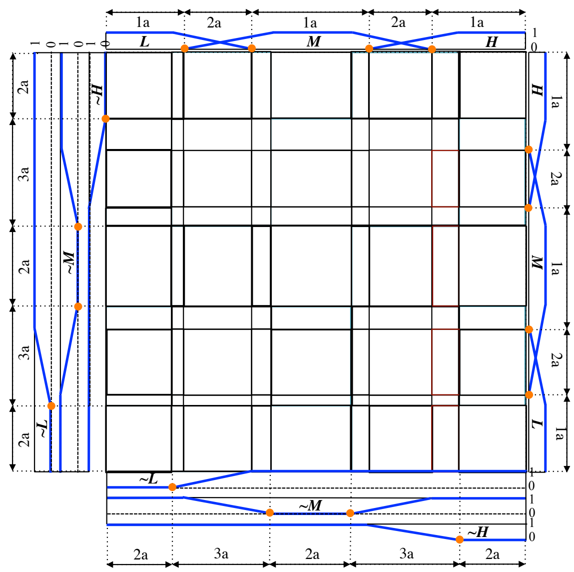


Fig. SM-8. Step 3.

4. Determine how many rules are fired (active) in each of the rectangles, as the sum of the MF and NMF rules that are fired in it. This is relatively easy to do from Fig. SM-8 because the MF and NMF labels have been put on both a horizontal and vertical axis. For, example, look at the top row of Fig. SM-9 and the third rectangle from the left, that is labeled "8". "8" was obtained as follows: For any point in this rectangle, look above and to the right to see that there are 2×1 MF rules that are active; look to the left and to the bottom to see that there are $2\times3=6$ NMF rules that are active; and, add the results to obtain $2+6=8$ active rules in that rectangle. This procedure has to be done 81 times. Although it is tedious to do, it is straightforward to do. Each of the 81 rectangles in Fig. SM-9 will now have a number in it. This is Fig. SM-9.

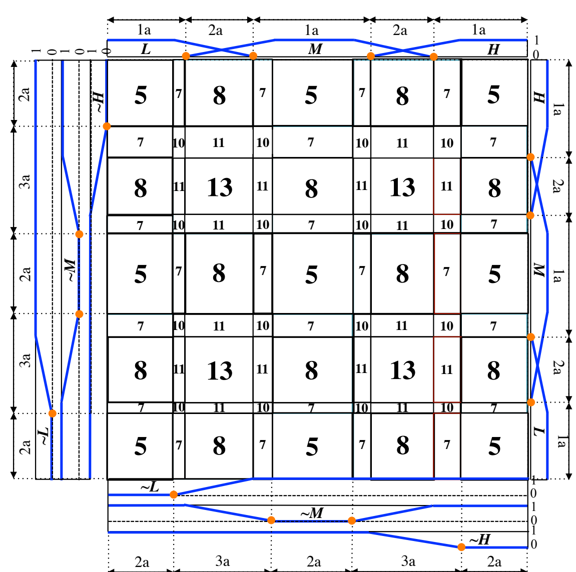


Fig. SM-9. Step 4.

5. Observing that many of the rectangles have the same number, it is a good idea to colorize them, so that rectangles with the same number of fired rules are easily located. First, copy all of the numbers from Fig. SM-9 onto another sheet, because when each rectangle is colorized its number gets put behind the rectangle. Instead of having to move each of the colored rectangles to the back one at a time (so that its number reappears) 81 times, after all of the rectangles have been colorized, the numbers from the other sheet can then be copied and located in one shot on top of all of the colorized rectangles. The result is Fig. 7, which is repeated below for the convenience of the reader.

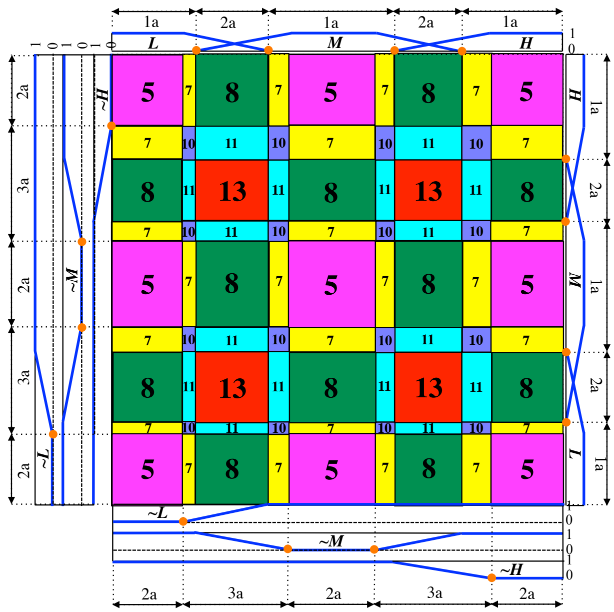
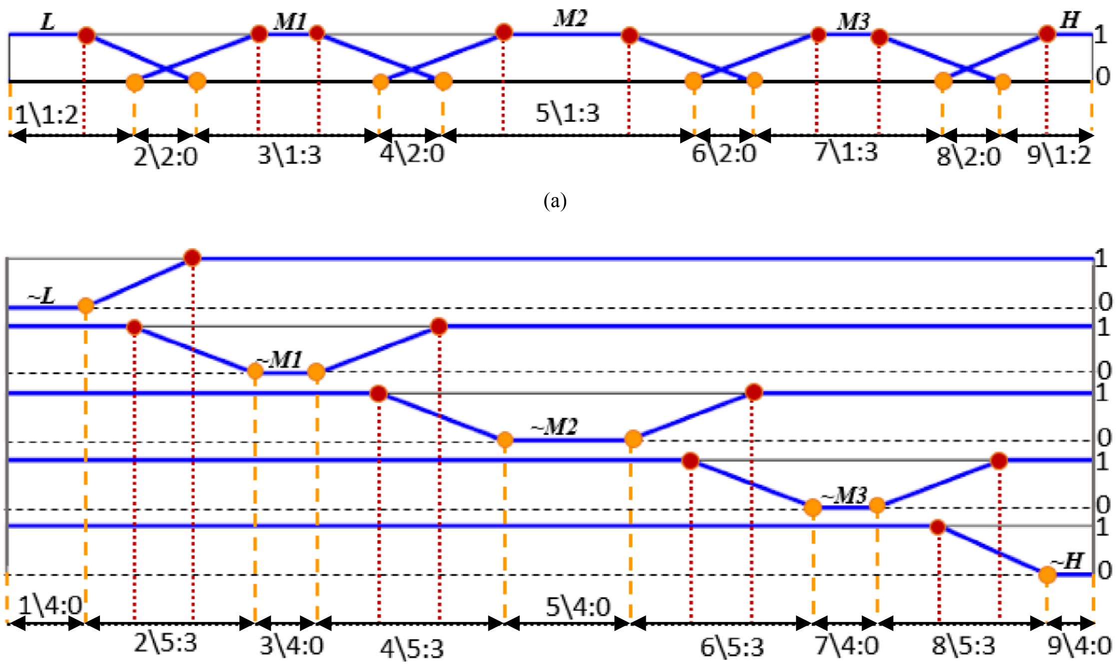


Fig. 7. Total number of first-order rule partitions in a two-variable intuitionistic fuzzy system and $Q = 3$.

IV. More Examples of Second-order MF and NMF Rule Partitions for One Variable

Example SM-5 (Continuation of Example SM-2): Figs. SM-10a,b began with Fig. SM-3 (MFs and NMFs) after which dotted vertical lines were drawn at grade-unity MF (NMF) kinks, indicated with brown filled-in circles. The second-order MF (NMF) rule partitions are drawn below the brown circles and extend into one row. There are 13 MF second-order rule partitions and 12 NMF second-order rule partitions. In this example (as in Example 10) the number of second-order NMF rule partitions is less than the number of second-order MF rule partitions.



(a)

(b)

Fig. SM-10. Second-order (a) MFs and (b) NMFs rule partitions for one variable, $Q = 5$.

Example SM-6 (Continuation of Example SM-3): Figs. SM-11a,b began with Fig. SM-4 (MFs and NMFs) after which dotted vertical lines were drawn at grade-unity MF (NMF) kinks, indicated with brown filled-in circles. The second-order MF (NMF) rule partitions are drawn below the brown circles and extend into one row. There are 7 MF second-order rule partitions and 8 NMF second-order rule partitions. In this example the number of second-order NMF rule partitions is greater than the number of second-order MF rule partitions.

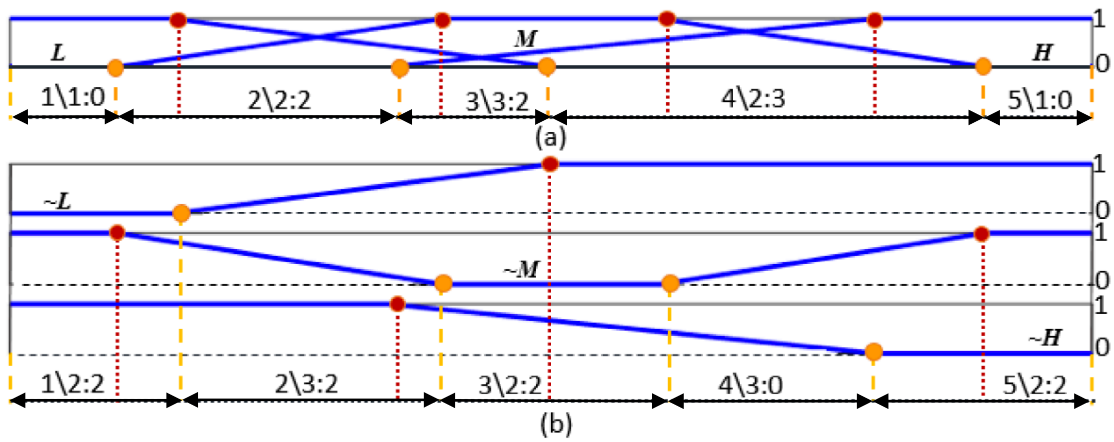


Fig. SM-11. Second-order (a) MFs and (b) NMFs rule partitions for one variable, $Q = 3$, when Assumption A is broken.

Example SM-7 (Continuation of Example SM-4): Figs. SM-12a,b began with Fig. SM-5 (MFs and NMFs) after which dotted vertical lines were drawn at grade-unity MF (NMF) kinks, indicated with brown filled-in circles. The second-order MF (NMF) rule partitions are drawn below the brown circles and extend into one row. There are 7 MF second-order rule partitions and 7 NMF second-order rule partitions. In this example the number of second-order NMF rule partitions is the same as the number of second-order MF rule partitions.

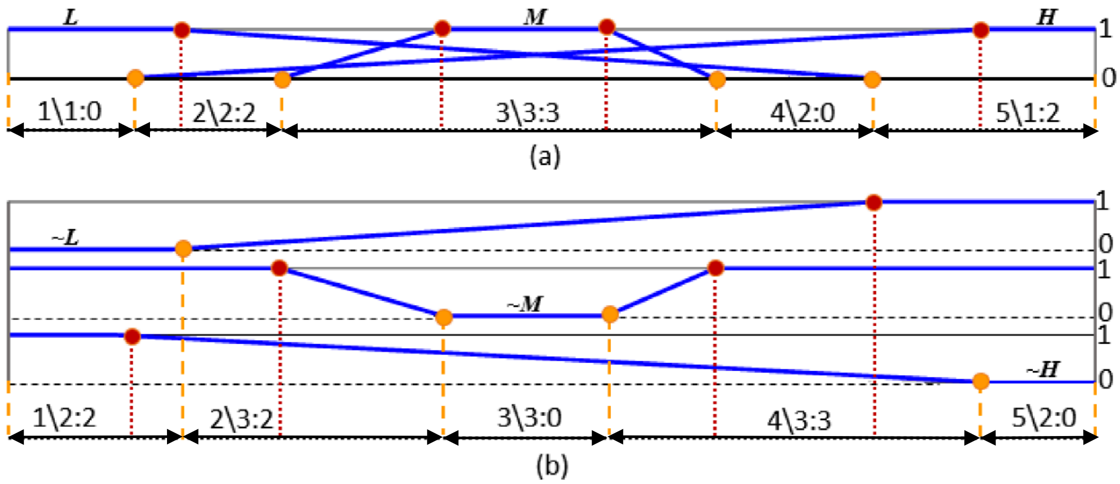


Fig. SM-12. Second-order (a) MFs and (b) NMFs rule partitions for one variable, $Q = 3$, when Assumption A is again broken.

V. Formulas for Computing the Total Number of Second-Order NMF and NMF Rule Partitions for p Variables

This section is taken from [1, Section III.B]. In Fig. SM-13 (which is Fig. 9 as it appears in [1, Fig. 6a]) observe that it is only in regions of $X_1 \times X_2$ where *both* x_1 and x_2 individually have no second-order rule partitions that such regions also have no second-order rule partitions. There are four such regions on Fig SM-13, two of which are $[a,b]\times[a,b]$ and $[a,b]\times[c,d]$. In such regions multiplying 0 by 0 gives the correct number of second-order rule partitions, which is also 0. On the other hand, regions of $X_1 \times X_2$ where either (but not both) x_1 or x_2 individually have no second-order rule partitions jointly have a *non-zero* number of second-order rule partitions. There are six such regions on Fig. SM-13, three of which are $[a,b]\times[0,a]$, $[a,b]\times[b,c]$ and $[c,d]\times[d,10]$. In such regions, multiplying 0 by any non-zero number always gives 0, which is not the correct number of the region's second-order rule partitions. If, instead the 0 is replaced by 1 then multiplying 1 by a non-zero number gives the correct number of the region's second-order rule partitions.

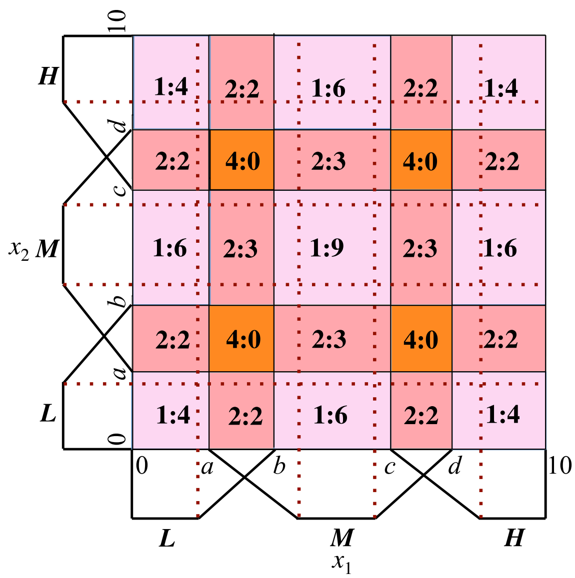


Fig. SM-13. Second-order MF rule partitions for two variables, $Q = 3$ [1].

These observations lead to the following novel way to compute $N^2(X_1, X_2)$ (see Table SM-IV for notation and definitions): Let the number of times that $(i = 1,2)$ $N^2(k_{x_i} \mid x_i) = 0$ be called $Z(X_i)$, and let

$$N^{2'}(X_i) \equiv N^2(X_i) + Z(X_i) \tag{SM-1}$$

Then

$$N^2(X_1, X_2) = N^{2'}(X_1)N^{2'}(X_2) - Z(X_1)Z(X_2) \tag{SM-2}$$

Example SM-8: Using Fig. 8a for both x_1 and x_2, observe that $N^2(X_1) = N^2(X_2) = 7$, and $Z(x_1) = Z(x_2) = 2$. Consequently, using (SM-1) and then (SM-2), one finds $N^{2'}(X_1) = N^{2'}(X_2)$ $= 7 + 2 = 9$ and $N^2(X_1, X_2)$ $= 9 \cdot 9 - 2 \cdot 2 = 77$, which agrees with $N^2(X_1, X_2)$ that was obtained in Example 12.

The extension of (SM-2) to p variables is:

$$N^2(X_1, X_2, \ldots, X_p) = \prod_{i=1}^{p} N^{2'}(X_i) - \prod_{i=1}^{p} Z(X_i) \tag{SM-3}$$

Equations (SM-1)–(SM-3) can also be used to compute the total number of second-order NMF rule partitions, using notations appropriate for them.

Example SM-9: Using Fig. 8b for both x_1 and x_2, observe that X_1 and X_2 both have six NMF second-order rule partitions, and that there are three NMF first-order rule partitions (1, 3 and 5) that have no second-order rule partitions. Consequently, for NMF's: $N^{2'}(X_1) = N^{2'}(X_2) = 6 + 3 = 9$ and $N^2(X_1, X_2)$ $= 9 \cdot 9 - 3 \cdot 3 = 72$, which agrees with $N^2(X_1, X_2)$ that was obtained in Example 13.

REFERENCE

[1] J. M. Mendel, "Explaining the performance potential of rule-based fuzzy systems as a *greater sculpting of the state space,*" *IEEE Trans. on Fuzzy Systems*, vol. 26, pp. 2362–2373, August 2018.