Comparing the Performance Potentials of Interval and General Type-2 Rule-Based Fuzzy Systems in Terms of *Sculpting the State Space*

Jerry M. Mendel, *Life Fellow, IEEE*

***Abstract*—This paper provides application-independent perspectives on why improved performance usually occurs as one goes from an interval type-2 (IT2) fuzzy system to a general type-2 (GT2) fuzzy system. This is achieved by using the horizontal-slice representation of a GT2 fuzzy set and GT2 fuzzy system and by examining first- and second-order rule partitions as well as novelty partitions for the horizontal slices. It demonstrates that, for triangle and trapezoid secondary membership functions, the numbers of first- and second-order rule partitions are exactly the *same* for IT2 and GT2 fuzzy systems, but that a maximum amount of change *always* occurs in *every* second-order rule partition of a GT2 fuzzy system. This does not always occur in such partitions of an IT2 fuzzy system. Furthermore, when type reduction (TR) is used in a GT2 fuzzy system, the total number of novelty partitions is directly proportional to the number of horizontal slices; consequently, there are many more such partitions in a GT2 fuzzy system that uses TR than occur in an IT2 fuzzy system that also uses TR. It is the author's conjecture that it is the maximum changes that occur in every second-order rule partition, as well as the greater number of novelty partitions when TR is used, that provide a GT2 fuzzy system with the potential to outperform an IT2 fuzzy system.**

***Index Terms*—General type-2 (GT2) fuzzy sets (FSs)/systems, horizontal slices, interval type-2 (IT2) fuzzy sets/systems, novelty partitions, rule-partitions, type reduction (TR).**

I. Introduction

In [32], it is stated that thousands of articles (including books) have been published about rule-based fuzzy systems (henceforth referred to as *fuzzy systems*), and "invariably they demonstrate that better performance [as measured by an application's performance metric(s)] is achieved by: (1) a type-1 (T1) fuzzy system over a nonfuzzy system, (2) an interval type-2 (IT2) fuzzy system over a T1 fuzzy system, and (3) a general type-2 (GT2) fuzzy system over an IT2 fuzzy system." That paper then raised the following crucial question: "*Why does improved performance occur as one goes from crisp to T1, to IT2, to GT2 fuzzy systems?*" It then went on to provide new and novel answers to this crucial question, outside of the context of a specific application (and so it represents a common component to all performance analyses), but only for T1 and IT2 fuzzy systems. The present paper extends the author's results in [32] to GT2 fuzzy systems, and, for the first time, lets one explain the performance potentials for T1, IT2, and GT2 fuzzy systems, outside of the context of a specific application.

To remind the reader, in [32], it was shown that: "... a T1 fuzzy system can sculpt its state space with greater variability than a crisp rule-based system can, and in ways that cannot be accomplished by the crisp system, and that an IT2 fuzzy system (that has the same number of rules as the T1 fuzzy system) can sculpt the state space with even greater variability, and in ways that cannot be accomplished by a T1 fuzzy system."

"Sculpting the state space" was quantified in [32] in terms of four kinds of partitions (see Appendix A for formal definitions).

1) *Uncertainty partitions* that let T1 fuzzy sets (FSs) be distinguished from crisps sets, IT2 FSs be distinguished from T1 FSs, and GT2 FSs be distinguished from IT2 FSs.
2) *First-order rule partitions* (each rule has p antecedents) that provide a course sculpting of $X_1 \times \cdots \times X_p$ into hyper-rectangles each of which contains the *same* number of the *same* fired rules.
3) *Second-order rule partitions* that provide a finer sculpting of $X_1 \times \cdots \times X_p$ when membership functions (MFs) change their mathematical formulae (slopes) within a first-order rule partition.
4) *Novelty partitions* that further sculpt $X_1 \times \cdots \times X_p$ in a novel but different way, when type reduction (TR) is used in an IT2 fuzzy system.

This paper re-examines both kinds of rule partitions and novelty partitions for GT2 fuzzy systems and, for the first time, shows that *they can sculpt the state space with even greater variability than can an IT2 fuzzy system, and in ways that cannot be achieved by an IT2 fuzzy system*.

Table VII summarizes everything that has been learned about crisp, T1, IT2, and GT2 fuzzy systems (herein and in [32]) in terms of first- and second-order rule partitions and novelty partitions; it provides the reader with the "big picture" and will help them communicate to others *why the potential for improved*

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The author is with the University of Southern California, Los Angeles, CA 90089-2564 USA, and also with the College of Artificial Intelligence, Tianjin Normal University, Tianjin 300384, China (e-mail: mendel@sipi.usc.edu).

This paper has supplementary downloadable multimedia material available at http://ieeexplore.ieee.org provided by the author. This includes 10 examples that illustrate how to construct alpha-plane FOUs, 10 examples that illustrate partitions for WH GT2 fuzzy systems, 10 examples that illustrate which MFs change or are unchanged in WH GT2 (alpha) second-order rule partitions (for alpha = 0, 0.5), and data for the WH GT2 FPID controller novelty partitions. This material is 8.76 MB in size.

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

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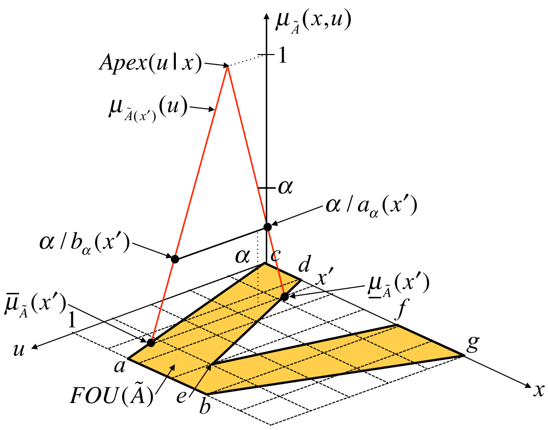

Fig. 1. Secondary set named $\tilde{A}(x')$, its triangular MF and $\alpha-$ cut raised to level α. Observe that $\mu_{\tilde{A}(x')}(u)$ is anchored to FOU$(\tilde{A})$ at $\underline{\mu}_{\tilde{A}}(x')$ and $\bar{\mu}_{\tilde{A}}(x')$ (see [31]).

performance exists as one goes from crisp to T1, to IT2, and then to GT2 fuzzy systems.

II. Background

A. General T2 Fuzzy Sets

As noted in [30, footnote 2]:

> In the early days of T2 fuzzy sets and systems, the phrases 'T2 FS' or 'T2 fuzzy system' were used in an all-inclusive way, meaning any kind of T2 FS or system. During the past 15 years [now more than 18 years] most of the attention has been given to IT2 FSs and systems. It is only within the past five [now more than eight] years or so that there has been a return to more general T2 FSs and systems, and, to distinguish them from the more specialized IT2 FSs and systems, the term 'general' is being used. In essence, T2 FSs and systems now consist of the union of interval and general T2 FSs and systems.

Because GT2 FSs are now en vogue (e.g., [2]–[20], [22]–[26], [29]–[31], [33], [35], [39]–[46], [49], [50], [52]–[55]), and many readers will not be as familiar with them as they are with IT2 FSs, they are summarized in Appendix B. It is very important for the reader to read that appendix, because many of its concepts are used below.

Due to space limitations, only parsimonious[1] triangle secondary MFs (see Fig. 1) are considered in this paper, but it is straightforward to obtain the same or very similar results for other piecewise-linear secondary MFs such as trapezoids.[2] For triangle secondary MFs $(w \in [0,1], \alpha \in [0,1])$

$$\begin{cases} \tilde{A}(x)_\alpha = [a_\alpha(x), b_\alpha(x)] \\ a_\alpha(x) = \underline{\mu}_{\tilde{A}}(x) + w[\bar{\mu}_{\tilde{A}}(x) - \underline{\mu}_{\tilde{A}}(x)]\alpha \\ b_\alpha(x) = \bar{\mu}_{\tilde{A}}(x) - (1-w)[\bar{\mu}_{\tilde{A}}(x) - \underline{\mu}_{\tilde{A}}(x)]\alpha \end{cases} \tag{1}$$

[1][31, p. 272] A MF of a T2 FS is said to be *parsimonious* when it is described by a small number of parameters. Parsimonious models have a very long history in the field of mathematical modeling, e.g., in system identification (e.g., [27]), one always tries to use a model with the fewest number of parameters to fit data. Because a T2 FS is a mathematical model, this author believes that parsimony should be adhered to for such models.

[2]Formulas for parsimonious symmetrical and nonsymmetrical trapezoid secondary MFs can be found in [31, p. 297].

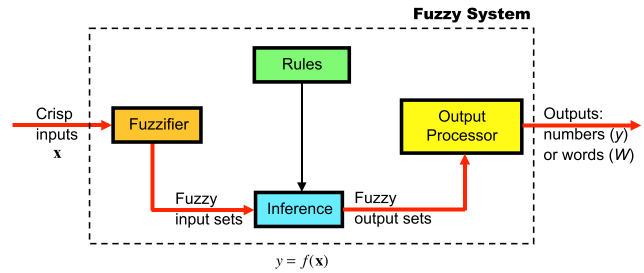

Fig. 2. Fuzzy system that is valid for IT2 and GT2 FSs (see [32]).

$$\text{Apex}(u|x) = \underline{\mu}_{\tilde{A}}(x) + w[\bar{\mu}_{\tilde{A}}(x) - \underline{\mu}_{\tilde{A}}(x)] \tag{2}$$

$$b_\alpha(x) - a_\alpha(x) = [\bar{\mu}_{\tilde{A}}(x) - \underline{\mu}_{\tilde{A}}(x)](1-\alpha). \tag{3}$$

Parameter w in (2) lets (1) include a wide range of triangle MFs, e.g., when $w = 0$ (or 1), the secondary MF is a right triangle that is perpendicular to $\underline{\mu}_{\tilde{A}}(x)$ (or $\bar{\mu}_{\tilde{A}}(x)$), and when $w = 0.5$, the secondary MF is an isosceles triangle. The length of the $\alpha-$cut in (3) depends only on α and not on w. For a trapezoid secondary MF, $b_\alpha(x) - a_\alpha(x)$ depends on both α and w.

Definition 1: The points at which the LMF, UMF, or secondary MF change its mathematical formula (slope) within the support of the *footprint of uncertainty* (FOU) are called *MF kinks*.

Example 1: For the FOU in Fig. 1, whose support is $[c, g] \times [0, 1]$, the UMF has two MF kinks (a, b) and the LMF has three MF kinks (d, e, f). The triangle secondary MF has a kink at its apex.

Because a secondary MF is a T1 FS, it can be represented using its $\alpha-$ cuts, as $\sup_{\alpha \in [0,1]} [\alpha / \tilde{A}(x)_\alpha]$, where $\tilde{A}(x)_\alpha$ is given in (B-17) (also, the top line of (1)). By connecting the $\alpha-$ cuts for $x \in X$, one obtains an α-plane for $\tilde{A}$, $\tilde{A}_\alpha$. When $\tilde{A}_\alpha$ is raised to level-α and the (fuzzy) union is taken with respect to $\alpha \in [0, 1]$, one obtains the *horizontal-slice* representation of $\tilde{A}$ that is given in (B-23). It is this representation that is used in our study of sculpting the state space for GT2 fuzzy systems.

B. GT2 Rule-Based Fuzzy Systems

For completeness, some of the background that is in [32, Sec. II] is repeated here, but in the context of GT2 fuzzy systems. A *rule-based GT2 fuzzy system* (*fuzzy system*, for short) contains four components—*rules*, *fuzzifier*, *inference (engine)*, and *output processor*—that are interconnected, as shown in Fig. 2. Once the rules have been established, the GT2 fuzzy system can again be viewed as a mapping from inputs to outputs, and this mapping can be expressed quantitatively as $y = f(\mathbf{x})$.

Because there are different ways to represent a GT2 FS (e.g., vertical-slice, wavy-slice, and horizontal-slice), there can be different ways to represent and implement a GT2 fuzzy system. To-date, only a horizontal-slice implementation has been developed, primarily because its calculations make use of the already well-understood calculations for an IT2 fuzzy system,

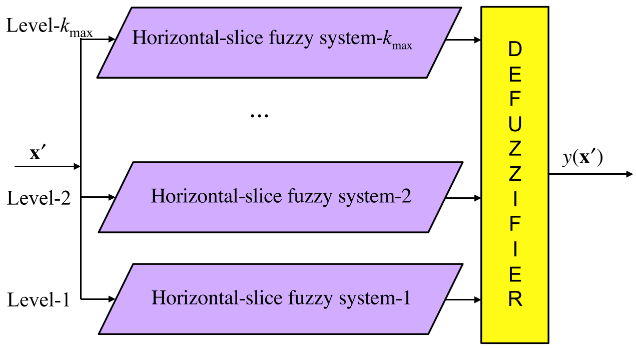

Fig. 3. WH GT2 fuzzy system is the aggregation of horizontal-slice IT2 fuzzy systems (see [31]).

and those calculations are used on each horizontal slice, as a result of the following fact ([53], [22]; see, also, [16], [17], and [38]): *$\alpha-$planes of a function of GT2 FSs equal that function applied to the $\alpha-$planes of those GT2 FSs.*[3]

Definition 2: A *horizontal-slice fuzzy system* is analogous to an IT2 fuzzy system where all of the well-known IT2 FS computations occur on the horizontal slice.

Definition 3: A *WH GT2 fuzzy system* is an aggregation of $k_{\max}$ horizontal-slice fuzzy systems, as in Fig. 3, where aggregation occurs by means of defuzzification.

The idea of aggregating horizontal-slice fuzzy systems was proposed originally by[4] Wagner and Hagras [44]–[46] and was expounded upon in [30]. As in [31, ch. 11], it is referred to in this paper as the *WH GT2* fuzzy system, so as to distinguish it from other kinds of GT2 fuzzy systems that may be developed in the future.

C. Rules

As in [32], it is assumed that a fuzzy system (IT2 or WH GT2) has M rules, p inputs $x_1 \in X_1, \ldots, x_p \in X_p$, and one output $y \in Y$, where x_i is described by Q_i linguistic terms $T_{x_i} = \{\tilde{X}_{ij}\}_{j=1}^{Q_i}$, and y is described either by Q_y linguistic terms, $T_y = \{\tilde{Y}_j\}_{j=1}^{Q_y}$, or by a function $g^l(x_1, \ldots, x_p)$ $(l = 1, \ldots, M)$. Just as the M rules of an IT2 fuzzy system can have two different canonical structures, Zadeh and TSK, the M rules of a WH GT2 fuzzy system can also have these two different structures. The distinction between IT2 and WH GT2 is associated with the nature of the MFs, which is not important when forming the rules. The structure of the rules remains exactly the same in the WH GT2 case but now some or all of the sets involved are GT2. Rule antecedents and consequents are still denoted $(i = 1, \ldots, p; l = 1, \ldots, M)$ $\tilde{F}_i^l$ and $\tilde{G}^l$, respectively. See [32, Def. 6] for the structures of the antecedents and consequents of generic Zadeh and TSK rules; this definition applies as well for GT2 FSs.

[3]This is a generalization of the well-known result for T1 fuzzy sets (since an $\alpha-$ plane is a union of $\alpha-$ cuts) that, under Zadeh's extension principle [51], $\alpha-$ cuts of a function of T1 FSs equal that function applied to the $\alpha-$ cuts of those T1 FSs (e.g., [21, Th. 2.9] and [37]).

[4]In the Wagner and Hagras references, the term "zSlice" is used instead of horizontal-slice.

D. Firing Set in a GT2 Fuzzy System

For a WH GT2 fuzzy system, fuzzy input sets in $X_1 \times \cdots \times X_p$, which flow through a set of M IF-THEN rules, are mapped into a GT2 fuzzy output set in Y, one horizontal slice at a time. The focus of this paper is primarily on the interaction of each fuzzy input with its respective GT2 antecedent, which then collectively lead to a[5] *T1 firing set* that is the same for both Mamdani and TSK WH GT2 fuzzy systems. In this paper, to keep things as simple as possible, we assume *singleton fuzzification*, which means that the "fuzzy input" is treated as a crisp number, although the approach that is taken herein is conceptually the same regardless of the nature of the fuzzifier.[6]

For each rule, it is known (e.g., [31, Th. 7.7]) that, when $x_i = x_i'$, only the vertical slice (Definition B.3) $\tilde{F}_i^l(x_i')$ of the rule-antecedent GT2 FS $\tilde{F}_i^l$ is activated, and it has the following $\alpha-\mathrm{cut}$ decomposition[7] [e.g., see (1) or (B-17) for $\alpha-$ cut notations] $(i = 1, \ldots, p; l = 1, \ldots, M)$:

$$\tilde{F}_i^l\left(x_i'\right) \Leftrightarrow \mu_{\tilde{F}_i^l\left(x_i'\right)}(u) = \sup_{\alpha\in[0,1]} \alpha \big/ \left[a_{i,\alpha}^l\left(x_i'\right), b_{i,\alpha}^l\left(x_i'\right)\right] \quad (4)$$

For a WH GT2 Mamdani or TSK fuzzy system, the *firing interval at level* α, $F_\alpha^l(\mathbf{x}')$, is $(l = 1, \ldots, M$ and[8] $\alpha \in [0,1] = 1/k_{\max}, 2/k_{\max}, \ldots, 1)$

$$F_\alpha^l\left(\mathbf{x}'\right) \equiv \left[\underline{f}_\alpha^l\left(\mathbf{x}'\right), \bar{f}_\alpha^l\left(\mathbf{x}'\right)\right] = \left[T_{i=1}^p a_{i,\alpha}^l\left(x_i'\right), T_{i=1}^p b_{i,\alpha}^l\left(x_i'\right)\right]. \quad (5)$$

In (5), T denotes a t-norm, usually the minimum or product.

Observe that in (5), for a WH GT2 fuzzy system, $\mathbf{x}'$ is processed nonlinearly $2k_{\max}$ times, $k_{\max}$ times using LMF quantities [the $a_{i,\alpha}^l(x_i')$], and $k_{\max}$ times using UMF quantities $[b_{i,\alpha}^l(x_i')]$. On the other hand, in an IT2 fuzzy system, $\mathbf{x}'$ is processed nonlinearly only twice using the LMF and UMF of FOUs.

Definition 4: In a WH GT2 fuzzy system, a T1 firing set is said to *contribute* to its output only if it is nonzero. It is the UMFs of rule antecedent $\alpha-$ planes that establish exactly where this occurs in $X_1 \times \cdots \times X_p$, and it occurs when the UMFs of rule antecedent $\alpha-$ planes $[b_{i,\alpha}^l(x_i')]$ are simultaneously nonzero[9] for $\alpha \in [0,1]$.

E. Type Reduction (TR) and Defuzzification

TR for a WH GT2 fuzzy system is performed for each horizontal slice fuzzy system after which the type-reduced results are aggregated across all of the horizontal slice fuzzy systems by means of defuzzification (see Fig. 3).

[5]In a T1 fuzzy system, this is a *firing level*, whereas in an IT2 fuzzy system, it is a *firing interval*.

[6]Nonsingleton fuzzification for a WH GT2 fuzzy system could include T1, IT2, or even GT2 nonsingleton fuzzifiers (see [31, Sec. 11.3] for further discussions about this).

[7]In a GT2 fuzzy system, one needs to keep track of (e.g., in $a_{i,\alpha}^l$) which antecedent is referred to (the first subscript *i*), which rule is referred to (the superscript *l*), and which $\alpha-\mathrm{cut}$ is referred to (the second subscript α); this unavoidably leads to heavy subscript and superscript notations.

[8]Why $\alpha = 0$ is not needed is explained below (8).

[9]If the UMF of a rule antecedent $\alpha-$ plane is zero, the LMF of a rule antecedent $\alpha-$ plane must also be zero because an LMF can never be larger than the UMF [i.e., in (1), $b_{i,\alpha}^l(x_i') \geq a_{i,\alpha}^l(x_i')$].

Definition 5: Horizontal-slice TR is TR applied to horizontal-slice quantities, the result being a *horizontal-slice type-reduced set*.

This paper focuses on horizontal-slice center-of sets TR (COS TR) followed by defuzzification, because COS TR is arguably the most widely used TR. However, the results in this paper are also applicable to height TR and centroid TR.

Definition 6: Horizontal-slice COS TR maps mixtures of the lower and upper values of the firing intervals at level α, in a WH GT2 Mamdani fuzzy system,[10] into $y_{l,\alpha}^{\mathrm{COS}}(\mathbf{x}')$ and $y_{r,\alpha}^{\mathrm{COS}}(\mathbf{x}')$, where $(Y_{\mathrm{COS},\alpha}(\mathbf{x}') = \alpha/[y_{l,\alpha}^{\mathrm{COS}}(\mathbf{x}'), y_{r,\alpha}^{\mathrm{COS}}(\mathbf{x}')])$:

$$y_{l,\alpha}^{\mathrm{COS}}(\mathbf{x}') = \frac{\sum_{i=1}^{L_\alpha} c_l(\tilde{G}_\alpha^i)\bar{f}_\alpha^i(\mathbf{x}') + \sum_{i=L_\alpha+1}^{M} c_l(\tilde{G}_\alpha^i)\underline{f}_\alpha^i(\mathbf{x}')}{\sum_{i=1}^{L_\alpha} \bar{f}_\alpha^i(\mathbf{x}') + \sum_{i=L_\alpha+1}^{M} \underline{f}_\alpha^i(\mathbf{x}')} \tag{6}$$

$$y_{r,\alpha}^{\mathrm{COS}}(\mathbf{x}') = \frac{\sum_{i=1}^{R_\alpha} c_r(\tilde{G}_\alpha^i)\underline{f}_\alpha^i(\mathbf{x}') + \sum_{i=R_\alpha+1}^{M} c_r(\tilde{G}_\alpha^i)\bar{f}_\alpha^i(\mathbf{x}')}{\sum_{i=1}^{R_\alpha} \underline{f}_\alpha^i(\mathbf{x}') + \sum_{i=R_\alpha+1}^{M} \bar{f}_\alpha^i(\mathbf{x}')} \tag{7}$$

in which $c_l(\tilde{G}_\alpha^i)$ and $c_r(\tilde{G}_\alpha^i)$ are the left and right end-points of the centroid of the horizontal slice of the GT2 rule-consequent $\tilde{G}^i$, and the *switch points* L_α and R_α have to be computed iteratively by any of the many different published algorithms, the most widely used being the KM [20], EKM [47], and EIASC [48].

There are different ways to perform defuzzification on any of the just-computed horizontal-slice type-reduced FSs (see [30] and [31, Sec. 11.7]). Here, for illustrative purposes, we use average of end-points defuzzification.

Definition 7: In *average of end-points defuzzification*, one first computes the average value of each $[y_{l,\alpha}^{\mathrm{COS}}(\mathbf{x}'), y_{r,\alpha}^{\mathrm{COS}}(\mathbf{x}')]$ and locates a spike at that value with amplitude α after which the k spikes are defuzzified by computing their COG, as

$$y_{\mathrm{WH}}(\mathbf{x}') = \frac{\sum_{i=1}^{k} \alpha_i \left[\left(y_{l,\alpha_i}^{\mathrm{COS}}(\mathbf{x}') + y_{r,\alpha_i}^{\mathrm{COS}}(\mathbf{x}')\right)\Big/ 2\right]}{\sum_{i=1}^{k} \alpha_i}. \tag{8}$$

Observe that $\alpha_i = 0$ makes no contribution in (8).[11]

F. First-Order Rule Partitions for T1 and IT2 Fuzzy Systems

One of the important things learned from [32] is that first- and second-order rule partitions of $X_1 \times \cdots \times X_p$ are completely determined by the respective rule partitions of each $(i = 1, \ldots, p)$ X_i separately, because when minimum or product t-norms are used, if even one component of a rule's firing level (interval) is zero, then that rule does not contribute to the output of the fuzzy system.[12] This carries over as well from T1 and IT2 fuzzy systems to WH GT2 fuzzy systems.

Definition 8: [32] In a T1 (IT2) fuzzy system, a *T1 (IT2) first-order rule partition* of X_i is a collection of nonoverlapping intervals in X_i, in each of which the *same* number of *same* rules is fired whose firing levels (intervals) contribute to the output of that system.

See [32, Tables II and III] as well as its Supplementary Materials for notations for T1 (IT2) first-order rule partitions; a formal *two-step procedure* for establishing these partitions on a drawing of the MFs (FOUs) of x_i; many examples of T1 and IT2 first-order rule partitions; and, formulas for the total number of T1 (IT2) first-order rule partitions of $X_1 \times \cdots \times X_p$, as well as for the fixed number of rules that are fired in each first-order rule partition. These formulas are provided below (see Section III-B) but in the context of horizontal slices. It was also observed in [32] that increasing the number of first-order rule partitions can be achieved by granulating x_i more finely into more FSs.

G. Second-Order Rule Partitions for T1 and IT2 Fuzzy Systems

Definition 9: [32] In a T1 (IT2) fuzzy system, a *T1 (IT2) second-order rule partition of* X_i occurs when the MF (FOU) of a T1 (IT2) FS that is associated with x_i changes its mathematical formula (slope) within a T1 (IT2) first-order rule partition of X_i.

Definition 10: The point at which an MF (LMF, UMF, or both) changes its mathematical formula (slope) *within (but not on the boundary of) a first-order rule partition* is called a *partition kink*.

See [32, Tables IV and V] as well as its Supplementary Materials for notations for T1 (IT2) second-order rule partitions; a formal *four-step procedure* for establishing these partitions on a drawing of the first-order rule partitions of X_i; many examples of T1 and IT2 second-order rule partitions; and, a formula for the total number of T1 (IT2) second-order rule partitions of $X_1 \times \cdots \times X_p$. This formula is provided below (see Section III-C) but in the context of horizontal slices. It was also observed in [32] that greater sculpting is achieved by using MFs (UMFs and LMFs) that have more partition kinks.

Finally, it was stated in [32] that

> Because an IT2 FS is described by two T1 FSs (LMF and UMF), it always has the potential to have more second-order partitions than a T1 FS; hence, an IT2 fuzzy system almost always has the potential to out-sculpt a T1 fuzzy system when both use the same number of MFs (FOUs) for each variable.[13] ... it is very common for an IT2 fuzzy system to have many (vastly) more second-order rule partitions than a T1 fuzzy system.

H. Novelty Partitions

Definition 11: [32] *IT2 novelty partitions* of $X_1 \times \cdots \times X_p$ occur only when TR is used and result from different end-points of a firing interval being used to compute the end-points of the type-reduced set. They occur within an IT2 first-order rule partition, regardless of whether or not there are any IT2 second-order rule partitions.

As is stated in [32]: IT2 novelty partitions

> ... provide us with additional insight into the further partitioning (sculpting) of $X_1 \times \cdots \times X_p$, something that can only occur for

[10]Similar equations occur for normalized A2-C0 and A2-C1 WH GT2 TSK fuzzy systems [31, Sec. 11.6.3].

[11]Wagner and Hagras [45] were the first to make this observation.

[12]For example, let f_i denote a firing level, then $\min(\text{any} f_i = 0, \text{all other} f_i) = 0$ and $\text{product}(\text{any} f_i = 0, \text{all other} f_i) = 0$.

[13]Why this is stated as "almost always" rather than as "always" is explained in [32, Sec. IV].

an IT2 fuzzy system that uses TR, and *can never occur in a T1 fuzzy system*. IT2 novelty partitions may help to explain why system performance is often arguably better for an IT2 fuzzy system that uses TR than it is for one that does not use TR.

III. Rule Partitions for WH GT2 Fuzzy Systems

According to Section II-F, the study of sculpting the state space for IT2 fuzzy systems begins by drawing the FOUs for the IT2 FSs of a generic variable $x_1 \in X_1$. This is very easy to do for IT2 FSs because an IT2 FS is defined by its FOU and it is easy to draw FOUs. In order to study the sculpting of the state space for WH GT2 fuzzy systems, we will need to draw $\alpha-$ plane FOUs. How to do this has (to the best knowledge of this author) never been explained before and is explained next for continuous functions of u.

A. Procedure for Drawing $\alpha-$ Plane FOUs

Here is an eight-step procedure for drawing $\alpha-$ plane FOUs[14] when both the LMF and UMF of an FOU are *piecewise linear.*[15]

1) Begin with the drawn $\alpha = 0$ FOUs.
2) If a point is shared by both the LMF and the UMF of an $\alpha = 0$ FOU, insert a star ($\star$) at that point [this may occur when $u = 0$ or $u = 1$; however, as a reviewer pointed out, this can also occur when the FOU contains a segment in which both the LMF and the UMF are the same function (e.g., a straight line), in which case the entire function is starred].
3) Locate all of the nonstarred MF kinks of the $\alpha = 0$ FOUs, and, if such an MF kink occurs on the LMF (UMF) of an $\alpha = 0$ FOU, insert a (green) vertical line extending from this kink to the upper (lower) MF of that FOU.
4) If the LMF and UMF of the left (right) end of the $\alpha = 0$ FOU do not touch, insert a (green) vertical line between them (e.g., see Example SM-1 in the Supplementary Material).
5) Choose the values of α for which α-plane FOUs are to be drawn, where $\alpha \in (0, 1]$. Call any one of the values α'. Repeat steps 6–8 for all of the chosen values of α.
6) Reposition each of the steps 3 and 4 vertical lines, by
   a) shortening it to length $b_{\alpha'}(x) - a_{\alpha'}(x)$ using (3) (this formula changes for other kinds of secondary MFs).
   b) moving it to position $b_{\alpha'}(x)$ using line three of (1) (this formula also changes for other kinds of secondary MFs).
   c) put a star on the top and bottom of these lines.
7) Create the $\alpha = \alpha'-$plane FOUs on the top of each of the $\alpha = 0$ FOUs by proceeding from the left to the right, and for each FOU
   a) connecting its top (bottom) starred points with straight lines, the result being the UMF (LMF) of the $\alpha = \alpha'-$plane FOU,[16]
   b) shading in the area between the UMF and LMF of the $\alpha = \alpha'$ FOU, the result being the $\alpha = \alpha'$-plane FOU (superimposed on the $\alpha = 0$ FOUs).
8) Remove the $\alpha = 0$ FOUs and all of the starred points, the result being the $\alpha = \alpha'-$plane FOUs.

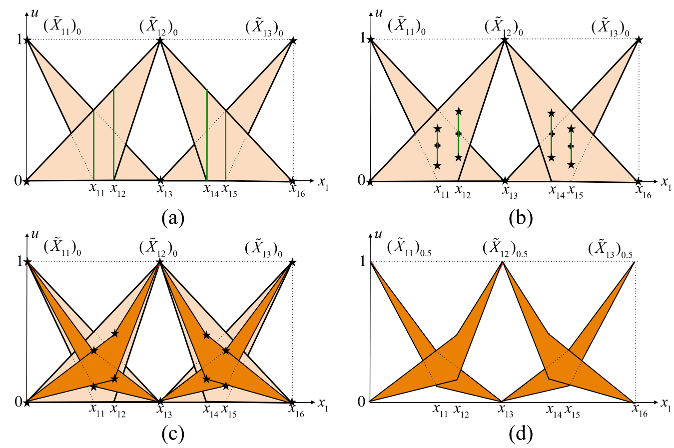

Fig. 4. Figures for Example 2. (a) FOUs and results of implementing: steps 1–4, (b) steps 5 and 6, (c) step 7, and (d) step 8.

Example 2: Consider x_1 described by three terms $\tilde{X}_{11}$, $\tilde{X}_{12}$, and $\tilde{X}_{13}$, each of whose FOU (step 1) is depicted in Fig. 4(a). Observe in Fig. 4(a) that (step 2) FOU$(\tilde{X}_{11})_0$ has a starred point at (0,1) and $(x_{13}, 0)$, FOU$(\tilde{X}_{12})_0$ has a starred point at (0,0), $(x_{13}, 1)$, and $(x_{16}, 0)$, and FOU$(\tilde{X}_{13})_0$ has a starred point at $(x_{13}, 0)$ and $(x_{16}, 1)$; (step 3) LMF$(\tilde{X}_{11})_0$ [UMF$(\tilde{X}_{11})_0$] has a nonstarred kink (green line) at $(x_{11}, 0)$ [nowhere], LMF$(\tilde{X}_{12})_0$ [UMF$(\tilde{X}_{12})_0$] has a nonstarred kink (green lines) at $(x_{12}, 0)$ and $(x_{14}, 0)$ [nowhere], and LMF$(\tilde{X}_{13})_0$[UMF$(\tilde{X}_{13})_0$] has a nonstarred kink (green line) at $(x_{15}, 0)$ [nowhere]. Step 4 is not activated.

For this example, the secondary MFs were assumed to be isosceles triangles ($w \equiv 0.5$) and (step 5) $\alpha' \equiv 0.5$, so that (step 6) the lengths of the green lines [see Fig. 4(b)] are shortened to[17] one-half of their [see Fig. 4(a)] lengths and the lines are centered[18] at the midpoints of the longer lines in Fig. 4(a). Fig. 4(c) and (d) depicts the results of steps 7 and 8, respectively.

Fig. 5 depicts the results of applying this eight-step procedure for four values of α'. Because the secondary MFs are triangles, the results for $\alpha' = 1$ are T1 FSs. A three-dimensional plot would show each of these four α' FOUs raised to its level α', but such a plot is not needed in the rest of this paper.

Example SM-1 in the Supplementary Material illustrates this eight-step procedure when the LMF and UMF of the left (right)

[14]Example 2 illustrates the steps of this procedure, and so the reader may want to read that example as he/she reads the steps of this procedure.

[15]The procedure for other kinds of LMFs and UMFs is explained at the end of this section.

[16]It is this step that makes use of the assumed piecewise linear natures of the LMF and UMF of the FOU, which lets adjacent starred points be connected by straight lines.

[17]Using (3), $b_{0.5}(x) - a_{0.5}(x) = 0.5[\bar{\mu}_{\tilde{A}}(x) - \underline{\mu}_{\tilde{A}}(x)]$.

[18]Using (1), $[b_{0.5}(x) + a_{0.5}(x)]/2 = [\bar{\mu}_{\tilde{A}}(x) + \underline{\mu}_{\tilde{A}}(x)]/2$.

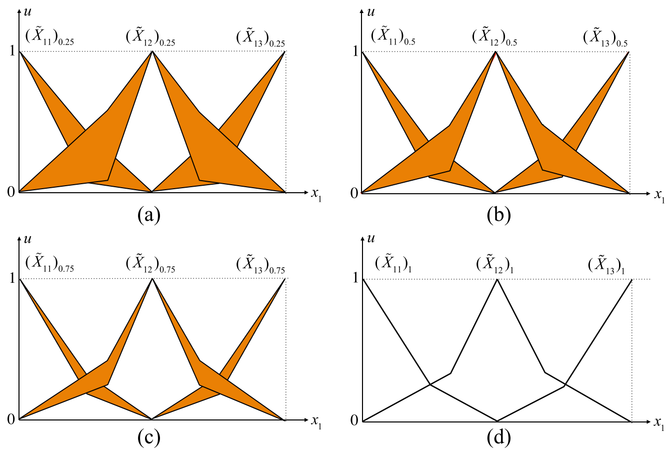

Fig. 5. Example 2 $\alpha'-$ planes, when $\alpha'=$ (a) 0.25, (b) 0.5, (c) 0.75, and (d) 1. For $\alpha' = 0$, use Fig. 4(a) (delete the four green vertical lines).

end of the $\alpha = 0$ FOU do not touch, so that step 4 is activated. Examples SM-2–SM-10 further illustrate this eight-step procedure.

When the LMF or the UMF of an FOU is not piecewise linear, then the simplest way to draw an α-plane FOU is to program $a_\alpha(x)$ and $b_\alpha(x)$ using (1) in which the nonlinear natures of $\bar{\mu}_{\tilde{A}}(x)$ and $\underline{\mu}_{\tilde{A}}(x)$ are inserted, and to let the computer provide the plot. Of course, this could also have been done when both the LMF and UMF of an FOU are piecewise linear, but this author believes that more insight is gained about rule partitions on each α-plane by using the eight-step procedure.

B. *First-Order Rule Partitions for WH GT2 Fuzzy Systems*

Definition 12: In a WH GT2 fuzzy system, a *WH GT2* (α) *first-order rule partition* of X_i is a collection of nonoverlapping intervals of X_i at level α, in each of which the *same* number of *same* rules is fired whose firing intervals at level α contribute to the output of that system.

Notations for WH GT2 (α) first-order rule partitions are given in Table I. A formal *two-step procedure*, which establishes $P^1_{\mathrm{GT2}(\alpha)}(k_i^\alpha|x_i)$, $N_R(k_i^\alpha)$, and $N^1_{\mathrm{GT2}(\alpha)}(X_i)$ on a drawing of the $\alpha-$ plane FOUs of x_i, is given in Table II.

Example 3: This is a continuation of Example 2. The results for step 1 in Table II are shown in Fig. 6 for five values of α'. Observe that $N^1_{\mathrm{GT2}(\alpha')}(X_i) = 2$ and that the locations and sizes of the WH GT2 (α') first-order rule partitions are the same for all five values of α'. Examples SM-11–SM-20, which are in the Supplementary Material, demonstrate similar results.

Formulas for $N^1_{\mathrm{GT2}(\alpha)}(X_1,\ldots,X_p)$ and $N_R(k_1^\alpha,\ldots,k_p^\alpha)$ are {these are analogous to [32, eqs. (6) and (7)]}

$$N^1_{\mathrm{GT2}(\alpha)}(X_1,\ldots,X_p) = \prod_{i=1}^{p} N^1_{\mathrm{GT2}(\alpha)}(X_i) \tag{9}$$

$$N_R\left(k_1^\alpha,\ldots,k_p^\alpha\right) = \prod_{i=1}^{p} N_R\left(k_i^\alpha\right). \tag{10}$$

From all of the examples, one observes that the number of WH GT2 (α) first-order rule partitions, as well as their locations

TABLE I
NOTATIONS USED FOR WH GT2 (α) FIRST-ORDER RULE PARTITIONS

First-Order Rule Partitions	
Symbol	Definition $(i=1,\ldots,p)$
$P^1_{GT2(\alpha)}(k_i^\alpha \mid x_i)$	WH GT2 (α) [a] *first-order* rule partition of X_i, numbered k_i^α
k_i^α	Counter/index[b] of WH GT2 (α) *first-order* rule partitions of X_i; $k_i^\alpha = 1,\ldots,N^1_{GT2(\alpha)}(X_i)$
$N^1_{GT2(\alpha)}(X_i)$	Total number of WH GT2 (α) *first-order* rule partitions of X_i (found by counting)
$N_R(k_i^\alpha)$	Fixed number of same rules fired in each $P^1_{GT2(\alpha)}(k_i^\alpha \mid x_i)$ (found by counting)
$P^1_{GT2(\alpha)}(k_1^\alpha,\ldots,k_p^\alpha)$	WH GT2 (α) *first-order* rule partition of $X_1\times\cdots\times X_p$, numbered $(k_1^\alpha,\ldots,k_p^\alpha)$
$N^1_{GT2(\alpha)}(X_1,\ldots,X_p)$	Total number of WH GT2 (α) *first-order* rule partitions of $X_1\times\cdots\times X_p$ [use (9)]
$N_R(k_1^\alpha,\ldots,k_p^\alpha)$	Fixed number of rules fired in each $P^1_{GT2(\alpha)}(k_1^\alpha, \ldots,k_p^\alpha)$ [use (10)]

[a] Subscript GT2 is short for WH GT2.
[b] Without loss of generality, k_{x_i} in [32] has been simplified to k_i.

TABLE II
TWO-STEP PROCEDURE FOR ESTABLISHING WH GT2 (α) FIRST-ORDER RULE PARTITION QUANTITIES FOR A SINGLE VARIABLE x_i ON A DRAWING OF ANY OF ITS α-PLANE FOUS

Step	Description
1	Scan the axis of x_i with an imaginary dashed vertical line from left to right. Count the number of intersections of this line with the α-plane FOUs of x_i; they represent the number of *same*-rules $[N_R(k_i^\alpha)]$ whose firing intervals at level α contribute to the output of a WH GT2 fuzzy system. When this number, or the nature of the same rules, changes draw a dashed vertical line; it represents the boundary of a WH GT2 (α) first-order rule partition. Insert a dashed vertical line at the start and at the end of X_i. For each x_i, the interval of real numbers between adjacent dashed vertical lines is its WH GT2 (α) first-order rule partition.
2	Count the number of $P^1_{GT2(\alpha)}(k_i^\alpha \mid x_i)$, the total being $N^1_{GT2(\alpha)}(X_i)$; then, $k_i^\alpha = 1,\ldots,N^1_{GT2(\alpha)}(X_i)$.

and sizes, is the same for all values of α. Two reasons for this are 1) starred points determined in step 2 of the eight-step procedure for drawing $\alpha-$plane FOUs are *fixed points* on all of those FOUs, because their vertical slice is a vertical line; and 2) these starred points establish locations of the boundaries of the WH GT2 (α) first-order rule partitions, and since they are fixed points, one *concludes* that *the number of WH GT2*(α) *first-order rule partitions, as well as their locations and sizes, is the same for all values of* α.

Because $\alpha = 0$ corresponds to the FOU of an IT2 fuzzy system (in which case, the results of this paper reduce to those in [32]), one can also conclude that *the number of first-order rule*

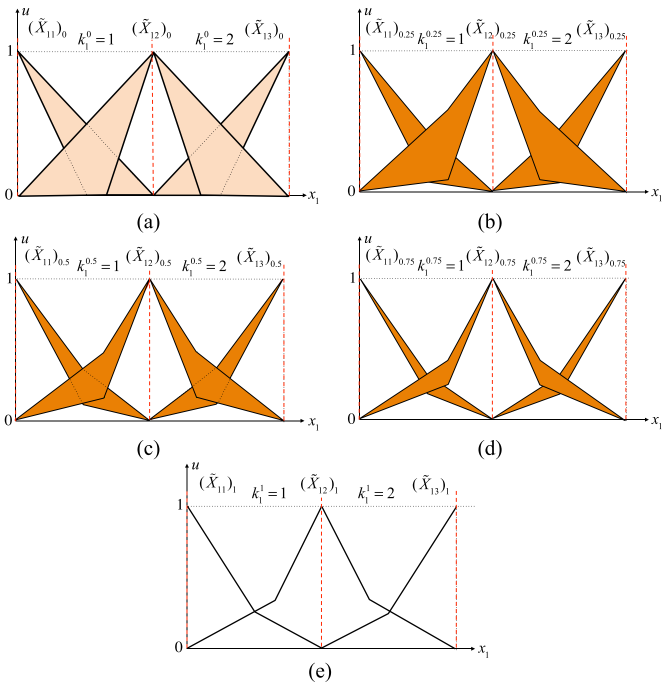

Fig. 6. Example 3 WH GT2 (α') first-order rule partitions when $\alpha'=$ (a) 0, (b) 0.25, (c) 0.5, (d) 0.75, and (e) 1.

partitions for a WH GT2 fuzzy system and its associated IT2 fuzzy system is the same.

C. Second-Order Rule Partitions for WH GT2 Fuzzy Systems

Definition 13: In a WH GT2 fuzzy system, a *WH GT2(α) second-order rule partition* of X_i occurs when any FOU at level α that is associated with x_i changes its mathematical formula (slope) within a WH GT2 (α) first-order rule partition of X_i.

Notations for WH GT2 (α) second-order rule partitions are given in Table III. A formal *four-step procedure* that establishes $P^2_{\mathrm{GT2}(\alpha)}(k_i^\alpha, m_{k_i^\alpha}|x_i)$, $N^2_{\mathrm{GT2}(\alpha)}(k_i^\alpha|x_i)$, and $N^2_{\mathrm{GT2}(\alpha)}(X_i)$ on a drawing of the WH GT2 (α) first-order rule partitions of X_i is given in Table IV.

Example 4: This is a continuation of Example 3. The results for steps 1–3 in Table IV are shown in Fig. 7 for five values of α'. Observe (see the encircled numbers) that, for all five values of α', $N^2_{\mathrm{GT2}(\alpha')}(1|x_1) = N^2_{\mathrm{GT2}(\alpha')}(2|x_1) = 3$, so that (step 4) $N^2_{\mathrm{GT2}(\alpha')}(X_1) = 6$. Additionally, observe that the locations and the widths of the WH GT2 (α') second-order rule partitions of X_i are the same for all five values of α'. Examples SM-11–SM-20, which are in the Supplementary Material, demonstrate similar results.

A formula for $N^2_{\mathrm{GT2}(\alpha)}(X_i)$ is

$$N^2_{\mathrm{GT2}(\alpha)}(X_i) = \sum_{k_i^\alpha=1}^{N^1_{\mathrm{GT2}(\alpha)}(X_i)} N^2_{\mathrm{GT2}(\alpha)}(k_i^\alpha|x_i). \tag{11}$$

TABLE III
NOTATION USED FOR WH GT2 (α) SECOND-ORDER RULE PARTITIONS

Second-Order Rule Partitions	
Symbol	Definition $(k_i^\alpha = 1,\ldots,N^1_{GT2(\alpha)}(X_i))$
$P^2_{GT2(\alpha)}(k_i^\alpha, m_{k_i^\alpha}\,\vert\, x_i)$	WH GT2 (α) *second-order* rule partition of X_i, in $P^1_{GT2(\alpha)}(k_i^\alpha\,\vert\, x_i)$, abbreviated to $(k_i^\alpha, m_{k_i^\alpha})$
$m_{k_i^\alpha}$	Counter/index of WH GT2 (α) *second-order* rule partition of X_i; $m_{k_i^\alpha} = 1,\ldots, N^2_{GT2(\alpha)}(k_i^\alpha\,\vert\, x_i)$
$N^2_{GT2(\alpha)}(k_i^\alpha\,\vert\, x_i)$	Total number of WH GT2 (α) *second-order* rule partitions in $P^1_{GT2(\alpha)}(k_i^\alpha\,\vert\, x_i)$ (found by counting)
$N^2_{GT2(\alpha)}(X_i)$	Total number of WH GT2 (α) *second-order* rule partitions of X_i [use (11)]
$P^2_{GT2(\alpha)}((k_1^\alpha,\ldots,k_p^\alpha), m_{(k_1^\alpha,\ldots,k_p^\alpha)})$	WH GT2 (α) *second-order* rule partition of $X_1\times\cdots\times X_p$ in $P^1_{GT2(\alpha)}(k_1^\alpha,\ldots,k_p^\alpha)$
$m_{(k_1^\alpha,\ldots,k_p^\alpha)} = 1,\ldots, N^2_{GT2(\alpha)}(k_1^\alpha,\ldots,k_p^\alpha)$	Counter/index of WH GT2 (α) *second-order* rule partition of $X_1\times\cdots\times X_p$
$N^2_{GT2(\alpha)}(k_1^\alpha,\ldots,k_p^\alpha)$	Total number of WH GT2 (α) *second-order* rule partitions in $P^1_{GT2(\alpha)}(k_1^\alpha,\ldots,k_p^\alpha)$
$N^2_{GT2(\alpha)}(X_1,\ldots,X_p)$	Total number of WH GT2 (α) *second-order* rule partitions of $X_1\times\cdots\times X_p$ [use (12)]

TABLE IV
FOUR-STEP PROCEDURE FOR ESTABLISHING WH GT2 (α) SECOND-ORDER RULE PARTITION QUANTITIES FOR A SINGLE VARIABLE x_i ON A DRAWING OF ITS RESPECTIVE WH GT2 (α) FIRST-ORDER RULE PARTITIONS

Step	Description
1	Scan the axis of x_i with an imaginary dotted vertical line from left to right. Wherever a LMF or UMF changes its formula, draw a dotted vertical line. If the change in formula occurs at a boundary of a WH GT2 (α) first-order rule partition, then do not draw such a vertical dotted line.
2	The interval of real numbers between adjacent dotted vertical lines or between a dotted line and a dashed (or dashed and dotted) line is its WH GT2 (α) second-order rule partition $P^2_{GT2(\alpha)}(k_i^\alpha, m_{k_i^\alpha}\,\vert\, x_i)$.
3	Each WH GT2 (α) first-order rule partition has from zero to a finite number of WH GT2 (α) second-order rule partitions; their count is $N^2_{GT2(\alpha)}(k_i^\alpha\,\vert\, x_i)$.
4	Count the total number of $N^2_{GT2(\alpha)}(k_i^\alpha\,\vert\, x_i)$, the total being $N^2_{GT2(\alpha)}(X_i)$ [use (11)].

In (11), $N^2_{\mathrm{GT2}(\alpha)}(k_i^\alpha|x_i)$ are obtained by counting (see Table IV, step 3). A formula for $N^2_{\mathrm{GT2}(\alpha)}(X_1,\ldots,X_p)$ is

$$N^2_{\mathrm{GT2}(\alpha)}(X_1,\ldots,X_p) = \prod_{j=1}^{p}\left[N^2_{\mathrm{GT2}(\alpha)}(X_j) + Z_\alpha(X_j)\right] - \prod_{j=1}^{p} Z_\alpha(X_j) \tag{12}$$

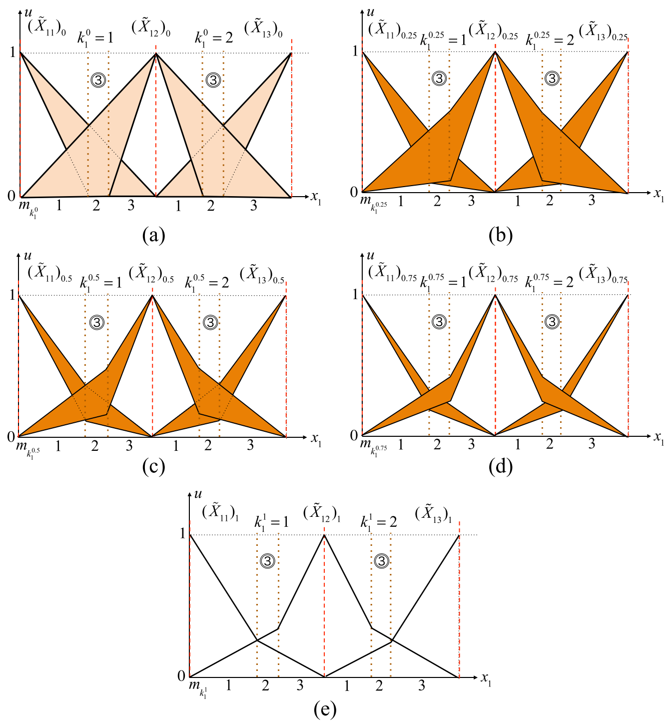

Fig. 7. Example 4 WH GT2 (α') first- and second-order rule partitions when $\alpha'=$ (a) 0, (b) 0.25, (c) 0.5, (d) 0.75, and (e) 1. The encircled numbers are $N^2_{\mathrm{GT2}(\alpha')}(k_1^{\alpha'}|x_1)$. The numbers along the horizontal axis $(m_{k_1}^{\alpha'})$ index the second-order rule partitions within their respective first-order rule partition.

$$Z_\alpha(X_j) = \sum_{k_j^\alpha=1}^{N^1_{\mathrm{GT2}(\alpha)}(X_j)} \xi\left(k_j^\alpha|x_j\right) \tag{13}$$

$$\xi(k_j^\alpha|x_j) = \begin{cases} 0 \text{ if } N^2_{\mathrm{GT2}(\alpha)}\left(k_j^\alpha|x_j\right) \neq 0 \\ 1 \text{ if } N^2_{\mathrm{GT2}(\alpha)}\left(k_j^\alpha|x_j\right) = 0 \end{cases}. \tag{14}$$

Note that (12) is analogous to [32, eq. (13)] and that the explanation and reason that are given for the appearance of $Z(X_j)$ in [32, eq. (13)] are the same for why $Z_\alpha(X_j)$ appears in our (12).

From all of the examples, one observes that the number of WH GT2 (α) second-order rule partitions, as well as their locations and sizes, is the same for all values of α. Three reasons for this are

1) Starred points determined in step 2 of the eight-step procedure for drawing $\alpha-$plane FOUs are *fixed points* on all of those FOUs, because their vertical slice is a vertical line.
2) Nonstarred MF kinks determined in step 3 of the eight-step procedure for drawing $\alpha-$plane FOUs are also *fixed locations* on all of those FOUs (the green vertical lines at those points are only repositioned vertically).
3) The starred points and the nonstarred MF kinks establish locations of the boundaries of the WH GT2 (α) second-order rule partitions, and, since they are fixed, one *concludes* that *the number of WH GT2*(α) *second-order rule partitions, as well as their locations and sizes, is the same for all values of* α.

Because $\alpha = 0$ corresponds to the FOU of an IT2 fuzzy system (in which case the results of this paper reduce to those in [32]), one also concludes that the *number of second-order rule partitions for a WH GT2 fuzzy system and its associated IT2 fuzzy system is the same.*

In summary, we have shown, somewhat surprisingly, that the *number, locations, and sizes of first- and second-order rule partitions for a WH GT2 fuzzy system and its associated IT2 fuzzy system are the same.*

D. Deeper Study of Second-Order Rule Partitions

The analyses that are given next demonstrate that *something fundamentally new occurs in a WH GT2 fuzzy system that lets it be distinguished from an IT2 fuzzy system.*

Changes of the mathematical formula (slope) of an FOU (Definition 13) can occur in any one of three different ways ($\tilde{A}$ is a generic T2 FS): 1) $\mathrm{LMF}(\tilde{A}_\alpha)$, 2) $\mathrm{UMF}(\tilde{A}_\alpha)$, or 3) both $\mathrm{LMF}(\tilde{A}_\alpha)$ and $\mathrm{UMF}(\tilde{A}_\alpha)$. Such changes were not examined in [32] but they are here.

Definition 14: As a transition is made from one WH GT2 (α) second-order rule partition of X_i to the next one, the next one is said to be *maximally changed* when both $\mathrm{LMF}(\tilde{A}_\alpha)$ and $\mathrm{UMF}(\tilde{A}_\alpha)$ change their formulas (slopes) in it; otherwise, when only $\mathrm{LMF}(\tilde{A}_\alpha)$ or $\mathrm{UMF}(\tilde{A}_\alpha)$ changes its formula (slope) in it, it is said to be *minimally changed*.

Example 5: This is a continuation of Example 4, but only for $\alpha' = 0$ and 0.5, since the numbers of WH GT2 (α') second-order rule partitions are the same for all values of α'. Examining Fig. 7(a), as x_1 sweeps from left to right, observe that

1) when $k_1^0 = 1$ [$(\tilde{X}_{11})_0$ and $(\tilde{X}_{12})_0$ are activated] and $m_{k_1^0}$ goes from 1 to 2 $(1 \rightarrow 2)$, $\mathrm{LMF}(\tilde{X}_{11})_0$ changes, and $\mathrm{UMF}(\tilde{X}_{11})_0$, $\mathrm{LMF}(\tilde{X}_{12})_0$, and $\mathrm{UMF}(\tilde{X}_{12})_0$ are unchanged;
2) when $k_1^0 = 1$ and $m_{k_1^0}$ goes from 2 to 3 $(2 \rightarrow 3)$, $\mathrm{LMF}(\tilde{X}_{11})_0$, $\mathrm{UMF}(\tilde{X}_{11})_0$, and $\mathrm{UMF}(\tilde{X}_{12})_0$ are all unchanged, but $\mathrm{LMF}(\tilde{X}_{12})_0$ changes;
3) when $k_1^0 = 2$ [$(\tilde{X}_{12})_0$ and $(\tilde{X}_{13})_0$ are activated] and $m_{k_1^0}$ goes from 1 to 2 $(1 \rightarrow 2)$, $\mathrm{LMF}(\tilde{X}_{12})_0$ changes, and $\mathrm{LMF}(\tilde{X}_{13})_0$, $\mathrm{UMF}(\tilde{X}_{12})_0$, and $\mathrm{UMF}(\tilde{X}_{13})_0$ are unchanged; and
4) when $k_1^0 = 2$ and $m_{k_1^0}$ goes from 2 to 3 $(2 \rightarrow 3)$, $\mathrm{LMF}(\tilde{X}_{12})_0$, $\mathrm{UMF}(\tilde{X}_{12})_0$, and $\mathrm{UMF}(\tilde{X}_{13})_0$ are all unchanged, but $\mathrm{LMF}(\tilde{X}_{13})_0$ changes. These four sets of results are summarized in the top portion in Table V.

Proceeding in the same manner for Fig. 7(c), one obtains the results that are summarized in the bottom portion in Table V. Observed in Table V that

1) the WH GT2(0.5) second-order rule partitions contain eight LMF or UMF changes, whereas the WH GT2(0) second-order rule partitions contain only four such changes, which is a 100% increase in MF changes for WH GT2(0.5) over WH GT2(0) (the IT2 fuzzy system);

TABLE V
WHICH MFS CHANGE (C) OR ARE UNCHANGED (UC) IN FIG. 7(A) AND (C) WH GT2 (α') SECOND-ORDER RULE PARTITIONS

$\alpha' = 0$ (Fig. 7a) WH GT2(0) fuzzy system: 4C's					
k_1^0	$m_{k_1^0}$	$\underline{(\tilde{X}_{11})_0}$	$\overline{(\tilde{X}_{11})_0}$	$\underline{(\tilde{X}_{12})_0}$	$\overline{(\tilde{X}_{12})_0}$
1	$1 \to 2$	**C**	UC	UC	UC
1	$2 \to 3$	UC	UC	**C**	UC
k_1^0	$m_{k_1^0}$	$\underline{(\tilde{X}_{12})_0}$	$\overline{(\tilde{X}_{12})_0}$	$\underline{(\tilde{X}_{13})_0}$	$\overline{(\tilde{X}_{13})_0}$
2	$1 \to 2$	**C**	UC	UC	UC
2	$2 \to 3$	UC	UC	**C**	UC
$\alpha' = 0.5$ (Fig. 7c) WH GT2(0.5) fuzzy system: 8C's					
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$\underline{(\tilde{X}_{11})_{0.5}}$	$\overline{(\tilde{X}_{11})_{0.5}}$	$\underline{(\tilde{X}_{12})_{0.5}}$	$\overline{(\tilde{X}_{12})_{0.5}}$
1	$1 \to 2$	**C**	**C**	UC	UC
1	$2 \to 3$	UC	UC	**C**	**C**
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$\underline{(\tilde{X}_{12})_{0.5}}$	$\overline{(\tilde{X}_{12})_{0.5}}$	$\underline{(\tilde{X}_{13})_{0.5}}$	$\overline{(\tilde{X}_{13})_{0.5}}$
2	$1 \to 2$	**C**	**C**	UC	UC
2	$2 \to 3$	UC	UC	**C**	**C**

TABLE VI
RULE BASE OF THE WH GT2 FPID CONTROLLER [32]

E / ΔE	$\tilde{N}$	$\tilde{Z}$	$\tilde{P}$
$\tilde{N}$	$\tilde{R}_Z^1 : U = -1$	$\tilde{R}_Z^2 : U = -0.5$	$\tilde{R}_Z^3 : U = 0$
$\tilde{Z}$	$\tilde{R}_Z^4 : U = -0.5$	$\tilde{R}_Z^5 : U = 0$	$\tilde{R}_Z^6 : U = 0.5$
$\tilde{P}$	$\tilde{R}_Z^7 : U = 0$	$\tilde{R}_Z^8 : U = 0.5$	$\tilde{R}_Z^9 : U = 1$

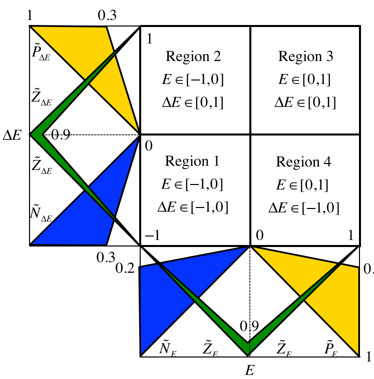

Fig. 8. FOUs and first-order rule partitions when $\alpha' = 0$ for the Example 6 WH GT2 FPID controller (see [32]).

2) the WH GT2(0.5) second-order rule partitions of X_i are always maximally changed, whereas (in this example), the WH GT2(0) second-order rule partitions of X_i are always minimally changed. *Finally, we are able to see some unique differences between the sculpting capabilities of WH GT2 and IT2 fuzzy systems.*

Examples SM-11–SM-20, which are in the Supplementary Material, further demonstrate similar results. Those ten examples demonstrate that

1) the WH GT2(0.5) second-order rule partitions of X_i are *always* maximally changed;
2) sometimes a WH GT2(0) second-order rule partition of X_i may be maximally changed, but most are not; and
3) except for Example SM-11, there is from a 33.33% to a 100% increase in MF changes for WH GT2(0.5) over WH GT2(0).

From all of these examples, one concludes that, even though the number of second-order rule partitions for a WH GT2 fuzzy system and its associated IT2 fuzzy system is the same, *the WH GT2 fuzzy system always has the maximum number of changes (two) in each of its second-order rule partitions, for each of its horizontal-slice fuzzy systems, whereas its associated IT2 fuzzy system does not.* It is this greater variability that provides the potential for better performance of a WH GT2 fuzzy system over an IT2 fuzzy system. Because this variability occurs within a second-order rule partition, it is still a second-order effect.

IV. NOVELTY PARTITIONS FOR WH GT2 FUZZY SYSTEMS

Paraphrasing [32, Sec. IV], so far our attention has been directed exclusively at the partitioning of $X_1 \times \cdots \times X_p$ due to the interactions of inputs to a WH GT2 fuzzy system with their respective antecedents. *Each horizontal slice second-order rule partition contains a nonlinear system where the exact nature of the nonlinearity depends on rule consequents and output processing*. In a WH GT2 fuzzy system that uses TR, there is another layer of partitioning of $X_1 \times \cdots \times X_p$ into *WH GT2*(α) *novelty partitions*.

Definition 15: WH GT2(α) *novelty partitions* of $X_1 \times \cdots \times X_p$ occur only when TR is used and result from different end-points of a horizontal level firing interval being used to compute the end-points of the type-reduced set. They occur within a WH GT2 (α) first-order rule partition, regardless of whether or not there are any WH GT2 (α) second-order rule partitions. WH GT2 (α) novelty partitions may help to explain why system performance is often arguably better for a WH GT2 fuzzy system that uses TR than it is for one that does not use TR.

It is very difficult to determine and display the geometry of WH GT2 (α) novelty partitions because there are no closed-form formulas for the end-points of a type-reduced set. So, as was done in [32, Sec. IV], we shall illustrate such partitions by means of an example.

Example 6: Our focus is on a WH GT2 fuzzy PID (FPID) controller U that has two normalized inputs, E and $\dot{E} \equiv \Delta E$. It uses the symmetrical 3×3 rule base in Table VI. The rule structure of the WH GT2 controller is $(l = 1, \ldots, 9)$

$$\tilde{R}_Z^l : \text{IF } E \text{ is } \tilde{F}_1^l \text{ and } \Delta E \text{ is } \tilde{F}_2^l \text{ THEN } U \text{ is } G^l. \quad (15)$$

In (15), both E and ΔE are described by three overlapping GT2 FSs whose FOUs are depicted in Fig. 8, and G^l are the crisp singletons that are tabulated in Table VI. The secondary MFs used here for all of the GT2 FSs are triangles. These GT2 FSs have been studied in Examples SM-1, SM-11, and SM-21 in the Supplementary Material. Example SM-11 shows that E and ΔE both have two WH GT2 (α) first-order rule partitions in which two rules are fired and no second-order rule partitions.

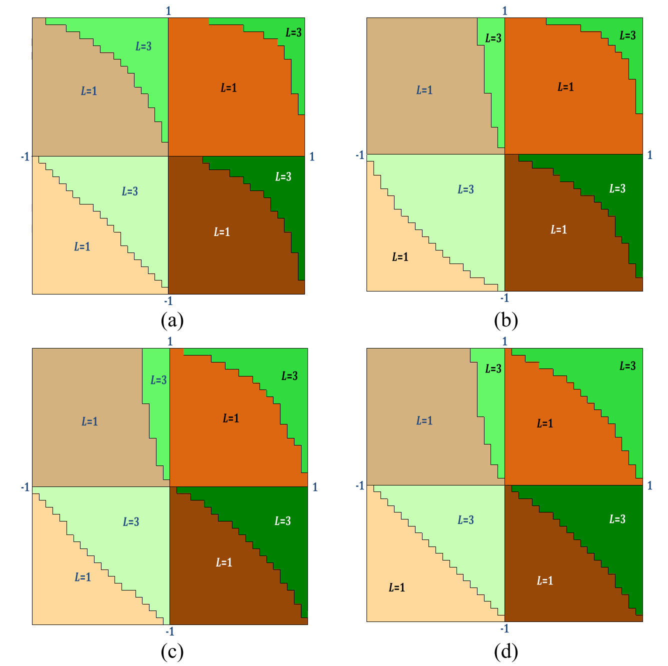

Fig. 9. WH GT2 (α) novelty partitions for $y_{l,\alpha}^{\mathrm{COS}}(E,\Delta E)$ in the WH GT2 FPID controller. Switch point L corresponds to L_α in (6), where (a) $\alpha = 0$ (the IT2 FPID controller), (b) $\alpha = 0.5$ and $w = 0$, (c) $\alpha = 0.5$ and $w = 0.5$, and (d) $\alpha = 0.5$ and $w = 1$.

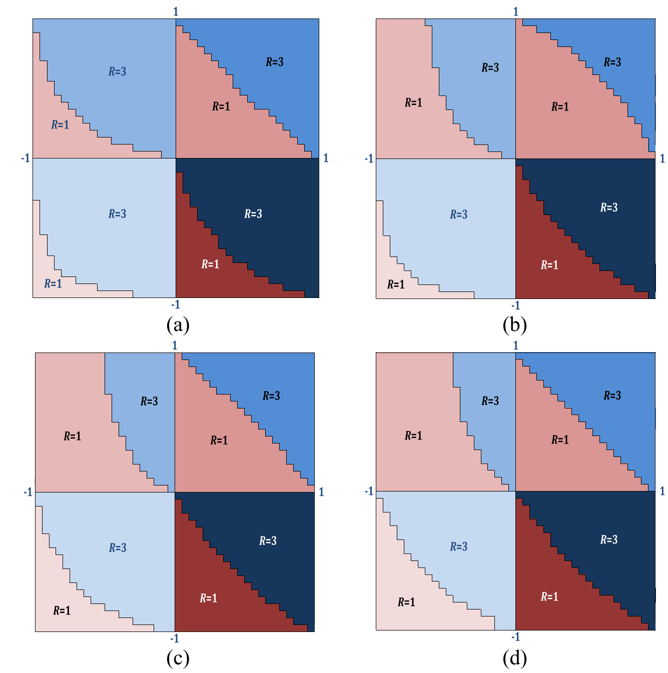

Fig. 10. WH GT2 (α) novelty partitions for $y_{r,\alpha}^{\mathrm{COS}}(E,\Delta E)$ in the WH GT2 FPID controller. Switch point R corresponds to R_α in (7), where (a) $\alpha = 0$ (the IT2 FPID controller), (b) $\alpha = 0.5$ and $w = 0$, (c) $\alpha = 0.5$ and $w = 0.5$, and (d) $\alpha = 0.5$ and $w = 1$.

Note also that when $\alpha' = 0$ the WH GT2 FPID controller is the same as the IT2 FPID controller.

As mentioned in [32], a T1 FPID controller that uses the UMFs of the three FOUs will also have two T1 first-order rule partitions in which two rules are fired, and no T1 second-order rule partitions. These facts make this a very interesting example, because the playing field has been leveled for T1, IT2, and WH GT2 FPID controllers in terms of first- and second-order rule partitions.

The main purpose of this example is to compare some WH GT2 (α) novelty partition plots of $y_{l,\alpha}^{\mathrm{COS}}(E,\Delta E)$ and $y_{r,\alpha}^{\mathrm{COS}}(E,\Delta E)$ for the four regions in Fig. 8, using three different triangle secondary MFs [in (1), $w = 0, 0.5, 1$] and two horizontal slices [$\alpha' = 0$ (the IT2 FPID controller) and $\alpha' = 0.5$].

Table SM-10 in the Supplementary Material tabulates important information for each of the four regions. This is such a relatively simple example that, instead of using any of the iterative algorithms to compute the switch points of the COS horizontal-slice type-reduced sets, it is more instructive to use brute force by considering the five possible iterations that are summarized in Table SM-11 in the Supplementary Material.

Figs. 9 and 10 depict the WH GT2 (α) novelty partitions, for $y_{l,\alpha}^{\mathrm{COS}}(E,\Delta E)$ and $y_{r,\alpha}^{\mathrm{COS}}(E,\Delta E)$, respectively, from which the following observations can be made.

1) Each of the four quadrants of $(E \times \Delta E)_\alpha$ has four novelty partitions.
2) WH GT2(0.5) novelty partitions for the three kinds of triangle secondary MFs all look quite different from their IT2 counterparts, thereby demonstrating that horizontal-slice TR leads to quite different novelty partitions.
3) The choice made for the kind of triangle secondary MF (by choosing different values of w) can significantly change the sizes of the GT2(0.5) novelty partitions.
4) There seems to be a flow in the sizes of the novelty partitions as w increases from 0 to 0.5 to 1.

When defuzzification is performed, using (8), each horizontal slice $(E \times \Delta E)_{\alpha_i}$ has eight WH GT2 (α_i) novelty partitions for both $y_{l,\alpha}^{\mathrm{COS}}(E,\Delta E)$ and $y_{r,\alpha}^{\mathrm{COS}}(E,\Delta E)$, for a total of 16 such novelty partitions. When $k_{\max}$ horizontal slices are used, (8) involves $16k_{\max}$ novelty partitions, whereas an IT2 FPID controller only involves 16 novelty partitions.

This scaling up of the number of novelty partitions is what provides the WH GT2 FPID controller that uses TR the *potential* for better performance than both the IT2 FPID controller and the WH GT2 FPID controller that does not use TR. How much improvement is actually obtained, what $k_{\max}$ should be, and what are the best locations for the $k_{\max}$ horizontal slices are topics for future research.

V. Summary

In order to see the forest from the trees, Table VII summarizes everything that has been learned to-date about crisp, T1, IT2, and GT2 fuzzy systems in terms of first- and second-order rule partitions, maximum changes in second-order rule partitions as well as novelty partitions. This table provides the reader with the "big picture" and should help them communicate to others

TABLE VII
SUMMARY[a]

Kind of system	First-order rule partitions (course sculpting)	Second-order rule partitions (fine sculpting)	Maximum changes in a second-order rule partition	Novelty partitions (further sculpting)
Crisp	Their number equals the number of MFs for each X_i Only one rule per partition	Not applicable, because a MF never changes its value of unity in a first-order partition	Not applicable, because this only applies to IT2 or WH GT2 fuzzy systems	Not applicable, because this only applies to IT2 and WH GT2 fuzzy systems that use type-reduction (TR)
Type-1 fuzzy	(Considerably) more of these than for a crisp system Because of overlap of MFs, there can be more than one rule per partition	Occur when MF changes its formula (slope) in a first-order partition	Not applicable, because this only applies to IT2 or WH GT2 fuzzy systems	Not applicable, because this only applies to IT2 and WH GT2 fuzzy systems that use TR
Interval type-2 fuzzy	Their number is the same as in a T1 fuzzy system Because of overlap of FOUs, there can be more than one rule per partition	Usually vastly more of these than in a T1 fuzzy system, because an FOU is described by two T1 FSs, its LMF and UMF	The formulas (slopes) for the LMF and UMF within a second-order rule partition may or may not both change; when they do, a maximum change occurs	Always occur when TR is used Provides further sculpting within first-order rule partitions, even if there are no second-order rule partitions
WH General type-2 fuzzy	Their number is the same, on each horizontal slice, as in T1 and IT2 fuzzy systems Because of overlap of horizontal-slice FOUs, there can be more than one rule per partition on each horizontal slice	Same number of these, on each horizontal slice, as in an IT2 fuzzy system	Always occurs, because on every horizontal slice the formulas (slopes) for the LMF and UMF within a second-order rule partition both change	Always occur on each horizontal slice when TR is used Provides further sculpting within first-order rule partitions, even if there are no second-order rule partitions Because there will be more than one horizontal slice there are many more of these than in an IT2 fuzzy system

[a] This table assumes singleton fuzzification and that the number of MFs, FOUs or GT2 FSs is the same for a specific X_i; that number can be different for other X_i.

why the potential for improved performance exists as one goes from crisp to T1, to IT2, and to GT2 fuzzy systems.

VI. Conclusions and Future Research

This paper has provided some new and novel application-independent perspectives on why improved performance usually occurs as one goes from an IT2 fuzzy system to a GT2 fuzzy system, and in so doing has extended the results given in [32] from IT2 fuzzy systems to GT2 fuzzy systems. It does this by using the horizontal-slice representation of a GT2 FS and GT2 fuzzy system (called a WH GT2 fuzzy system) and by examining first- and second-order rule partitions, as well as novelty partitions for the horizontal slices. It has demonstrated that, for triangle and trapezoid secondary MFs, the numbers of first- and second-order rule partitions are the *same* for IT2 and WH GT2 fuzzy systems, but that a maximum amount of change occurs in *every* second-order rule partition for each of the horizontal-slice fuzzy systems of a WH GT2 fuzzy system, whereas this does not occur in such partitions in an IT2 fuzzy system. It has also demonstrated that when TR is used in a WH GT2 fuzzy system, the total number of novelty partitions is directly proportional to the number of horizontal slices, and that consequently, there are many more such partitions that occur in an IT2 fuzzy system that also uses TR. Table VII presents the reader with the “big picture” and should help them communicate to others *why the potential for improved performance exists as one goes from crisp to T1, to IT2, and to GT2 fuzzy systems*.

It is the author’s conjecture that it is the maximum changes that occur in every second-order rule partition, as well as the greater number of novelty partitions, when TR is used that provides a GT2 fuzzy system with the potential to outperform an IT2 fuzzy system.

Some open research questions and extensions to this paper are

1) Extend its results to (a) nonsingleton fuzzification, (b) piecewise-continuous and convex secondary MFs, and (c) Gaussian FOUs;
2) Study how (or if) first- and second-order partitions, as well as novelty partitions, can be extended to nonconvex secondary MFs [40], [49];
3) Examine the robustness of the defuzzified output of a WH GT2 fuzzy system to the number and locations of horizontal slices [I conjecture that two or three horizontal slices will suffice, because of the clustering effect that is present in the defuzzification formula (8)];
4) Quantify the difference between the defuzzified outputs of IT2 and a WH GT2 fuzzy systems; and
5) Examine the following conjecture: the switch points L_α and R_α, in (6) and (7). respectively, remain fixed within a second-order rule partition.

Appendix A
Background on Uncertainty Partitions[19]

Definition A.1: A *crisp partition* [see Fig. 11(a)] of the real variable x comprises nonoverlapping adjacent regions that are intervals of real numbers, where the degree of membership in each region is 1. They can be described mathematically using classical (crisp) sets.

Definition A.2: A *first-order uncertainty partition* [see Fig. 11(b)] of the real variable x comprises overlapping intervals, where one is absolutely certain about where the overlap begins and ends, so that the degree of membership in each

[19]The material in this appendix has been taken from [32].

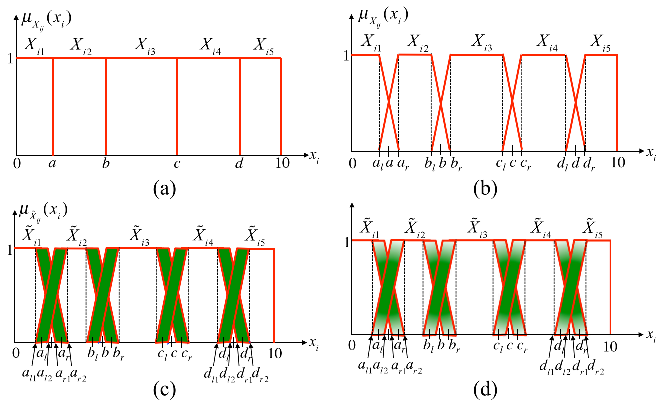

Fig. 11. Four kinds of partitions. (a) Crisp. (b) First-order uncertainty. (c) Second-order uncertainty with uniform weighting. (d) Second-order uncertainty with nonuniform weighting. (see [32]).

region of overlap is a real number that is an element of [0,1]. They can be described mathematically using classical (T1) FSs.

Definition A.3: A *second-order uncertainty partition* [see Fig. 11(c)] of the real-variable x comprises overlapping intervals where one is unsure about where the overlap begins and ends, so that the degree of membership in each region of overlap is an interval of real numbers that is a subset of $[0, 1]$.

Definition A.4: Each region in $X_i \times [0, 1]$, in which the degree of membership is an interval of real numbers, is called the *footprint of uncertainty* (FOU) of $\tilde{X}_{ij}$ [28], [31].

Definition A.5: A uniformly (nonuniformly) shaded FOU, as in Fig. 11(c) and (d) denotes a uniform (nonuniform) weighting of all of its points and is called a *uniformly* (*nonuniformly*) *weighted second-order uncertainty partition*. Uniformly (nonuniformly) weighted partitions can be described mathematically using IT2 (GT2) FSs.

Appendix B
Background on General Type-2 Fuzzy Sets[20]

Definition B.1: A *type-2 fuzzy set* (T2 FS; also called a *general T2 FS*), denoted $\tilde{A}$, is the graph of a bivariate function [1]—called the MF of $\tilde{A}$—on the Cartesian product $X \times [0, 1]$ into $[0, 1]$, where X is the universe for the *primary variable* of $\tilde{A}$, x. The MF of $\tilde{A}$ is denoted $\mu_{\tilde{A}}(x, u)$ ($\mu_{\tilde{A}}$ for short), called a *type-2 MF*, i.e.,

$$\tilde{A} = \{((x, u), \mu_{\tilde{A}}(x, u)) \,|\, x \in X, u \in U \equiv [0, 1]\} \tag{B-1}$$

in which $0 \leq \mu_{\tilde{A}}(x, u) \leq 1$. U is the universe for the *secondary variable* u, and in this paper U is always assumed to be $[0, 1]$. $\tilde{A}$ can also be expressed (using Zadeh's FS notation), as

$$\tilde{A} = \int_{x \in X} \int_{u \in [0,1]} \mu_{\tilde{A}}(x, u)/(x, u) \tag{B-2}$$

where $\int\int$ denotes union over all admissible x and *u*.[21]

[20]The material in this Appendix has been taken from [31, ch. 6].

[21]For discrete universes of discourse, in (B-2) $\int$ is replaced by $\sum$, X is replaced by X_d, and $[0, 1]$ is replaced by $\{0, u_1, u_2, \ldots, u_{n-1}, 1\}$.

Definition B.2: For every $x \in X$, the value of $\mu_{\tilde{A}}(x, u)$, $f_x(u)$, is called the *secondary grade* of x; hence, if $x \in X$, then $f_x(u) \equiv \mu_{\tilde{A}}(x, u)$, where $0 \leq f_x(u) \leq 1$.

Definition B.3: A *secondary MF*, $\mu_{\tilde{A}(x)}(u)$, is [1] a restriction of function $\mu_{\tilde{A}} : X \times [0, 1] \to [0, 1]$ to $x \in X$, i.e., $\mu_{\tilde{A}(x)} : [0, 1] \to [0, 1]$, or in FS notation

$$\mu_{\tilde{A}(x)}(u) = \int_{u \in [0,1]} \mu_{\tilde{A}}(x, u)/u = \int_{u \in [0,1]} f_x(u)/u. \tag{B-3}$$

Note, importantly, that $\tilde{A}(x)$ is a T1 FS,[22] which is also referred to as a *secondary set*, and as such it can be represented by its $\alpha-$ cut decomposition. $\{\mu_{\tilde{A}(x)}(u) | u \in [0, 1]\}$ is also called a *vertical slice* of $\mu_{\tilde{A}}(x, u)$ (see, also, Definition B.10).

Definition B.4: J_x, the *primary membership* of *x*, is[23]

$$J_x = \{(x, u) | u \in [0, 1], \mu_{\tilde{A}}(x, u) > 0\}. \tag{B-4}$$

It can also be expressed as [36] a subset of $\{x\} \times I_x$, i.e.,

$$J_x = \{x\} \times I_x \tag{B-5}$$

where

$$I_x = \{u \in [0, 1] | \mu_{\tilde{A}}(x, u) > 0\}. \tag{B-6}$$

I_x is the support of the secondary MF and can be *connected*[24] or *disconnected*.

Definition B.5: [36] The *support* of $\tilde{A}$ [1] comprises all $(x, u) \in X \times [0, 1]$ such that $\mu_{\tilde{A}}(x, u) > 0$ and is also called the *domain of uncertainty* of $\tilde{A}$, DOU($\tilde{A}$), i.e.,

$$\text{DOU}(\tilde{A}) = \{(x, u) \in X \times [0, 1] | \mu_{\tilde{A}}(x, u) > 0\} = \bigcup_{x \in X} J_x. \tag{B-7}$$

Definition B.6: When the support of the secondary MF, I_x, is *closed* (so that it is connected; see footnote 24) for $x \in X$, i.e.,

$$I_x = \{u \in [0, 1] | \mu_{\tilde{A}}(x, u) > 0\} = \left[\underline{\mu}_{\tilde{A}}(x), \bar{\mu}_{\tilde{A}}(x)\right] \tag{B-8}$$

where [1]

$$\bar{\mu}_{\tilde{A}}(x) = \sup \{u | u \in [0, 1], \mu_{\tilde{A}}(x, u) > 0\} \tag{B-9}$$

$$\underline{\mu}_{\tilde{A}}(x) = \inf \{u | u \in [0, 1], \mu_{\tilde{A}}(x, u) > 0\} \tag{B-10}$$

then the domain of uncertainty of $\tilde{A}$ is called the *footprint of uncertainty* of $\tilde{A}$, FOU($\tilde{A}$), i.e.,

$$\begin{aligned}\text{DOU}(\tilde{A}) &= \text{FOU}(\tilde{A}) \\ &= \left\{(x, u) | x \in X \text{ and } u \in \left[\underline{\mu}_{\tilde{A}}(x), \bar{\mu}_{\tilde{A}}(x)\right]\right\}.\end{aligned} \tag{B-11}$$

[22]Notation $\mu_{\tilde{A}(x)}(u)$ (which is different from the notation used in [28]) is consistent with the usual labeling of an MF for a T1 FS, where $\tilde{A}(x)$ is the name of that set.

[23]In [28] and much of the type-2 literature, it is stated that $J_x \subseteq [0, 1]$ and J_x is left undefined, both of which have been very problematic. In this paper, the statement $J_x \subseteq [0, 1]$ has been abandoned and J_x is defined. For additional discussions about this, see [31, Sec. 6.6] and [36].

[24]A set $A \subseteq \mathbb{R}$ is *connected* if and only if A is an interval (closed, open, or neither).

Note that $\underline{\mu}_{\tilde{A}}(x)$ and $\bar{\mu}_{\tilde{A}}(x)$ are called the *lower* and *upper MFs* of FOU$(\tilde{A})$ [34] and are the lower and upper (T1 FS) bounding functions of the FOU, respectively. Commonly used abbreviations for the lower and upper MFs of $\tilde{A}$ are $\mathrm{LMF}(\tilde{A})$ and $\mathrm{UMF}(\tilde{A})$.

Definition B.7: The *support* of $\mathrm{LMF}(\tilde{A})$ $[\mathrm{UMF}(\tilde{A})]$ is the crisp set of all points $x \in X$ such that $\underline{\mu}_{\tilde{A}}(x) > 0$ $[\bar{\mu}_{\tilde{A}}(x) > 0]$.

Definition B.8: When $u \in [0,1]$ and $\mu_{\tilde{A}}(x,u) = 1$ for $x \in X$, then $\tilde{A}$ is called an *interval T2 FS* (*IT2 FS*); it is completely described by its DOU, so that

$$\tilde{A} = 1/\mathrm{DOU}(\tilde{A}). \tag{B-12}$$

Equation (B-12) is an expressive equation that means $\mu_{\tilde{A}}(x, u) = 1$ for $(x,u) \in \mathrm{DOU}(\tilde{A})$, where $\mathrm{DOU}(\tilde{A})$ is given in (B-7) in which $\mu_{\tilde{A}}(x,u) > 0$ is replaced by $\mu_{\tilde{A}}(x,u) = 1$.

Definition B.9 [36]: An IT2 FS is called a *closed IT2 FS* (CIT2 FS) when I_x is closed for $x \in X$ (see Definition B.6). In this case, $\mathrm{DOU}(\tilde{A}) = \mathrm{FOU}(\tilde{A})$; hence, for a CIT2 FS, (B-12) can be expressed as

$$\tilde{A} = 1/\mathrm{FOU}(\tilde{A}) \tag{B-13}$$

where $\mathrm{FOU}(\tilde{A})$ is defined in (B-11). Another way to express $\mathrm{FOU}(\tilde{A})$ for a CIT2 FS is [see, also, (B-7)]

$$\mathrm{FOU}(\tilde{A}) = \bigcup_{x\in X} \{x\} \times I_x = \bigcup_{x\in X} J_x. \tag{B-14}$$

CIT2 has often been shortened to IT2.

Definition B.10: The *vertical-slice representation* of GT2 FS $\tilde{A}$ focuses on each value of the primary variable x and expresses (B-1) as the union of all of its secondary T1 FSs, i.e.,

$$\tilde{A} = \int_{x\in X} \mu_{\tilde{A}(x)}(u)/x \tag{B-15}$$

so that [30]

$$\begin{aligned}\tilde{A} &= \int_{x\in X} \left[\bigcup_{\alpha\in[0,1]} \left[\alpha/\tilde{A}(x)_\alpha\right]\right] \Big/ x \\ &= \int_{x\in X} \sup_{\alpha\in[0,1]} \left[\alpha/\tilde{A}(x)_\alpha\right] \Big/ x\end{aligned} \tag{B-16}$$

in which the $\alpha-$ cut of the T1 FS $\tilde{A}(x)$, $\tilde{A}(x)_\alpha$, is given by

$$\tilde{A}(x)_\alpha = \left\{u \,\middle|\, \mu_{\tilde{A}(x)}\ (u) \geq \alpha\right\} \equiv [a_\alpha(x), b_\alpha(x)]\,. \tag{B-17}$$

Definition B.11: An *α-plane* [26], [35] for a GT2 FS $\tilde{A}$, denoted $\tilde{A}_\alpha$, is the union of all primary memberships of $\tilde{A}$ whose secondary grades are greater than or equal to $\alpha \in [0,1]$, i.e.,

$$\begin{aligned}\tilde{A}_\alpha &= \{(x,u), \mu_{\tilde{A}}(x,u) \geq \alpha \,|x \in X, u\in[0,1]\,\} \\ &= \int_{x\in X}\int_{u\in[0,1]} \{(x,u)\,|\,f_x(u) \geq \alpha\}.\end{aligned} \tag{B-18}$$

Alternatively, $\tilde{A}_\alpha$ can be expressed by means of (B-17), as

$$\tilde{A}_\alpha = \int_{x\in X} \tilde{A}(x)_\alpha/x = \int_{x\in X} [a_\alpha(x), b_\alpha(x)]/x. \tag{B-19}$$

$\tilde{A}_\alpha$ has an LMF and a UMF, where $(x \in X)$

$$\begin{cases}\mathrm{LMF}(\tilde{A}_\alpha) = a_\alpha(x) \\ \mathrm{UMF}(\tilde{A}_\alpha) = b_\alpha(x)\end{cases}. \tag{B-20}$$

When an α-plane is raised to level α, one obtains a *horizontal slice*[25] *at level* α, $R_{\tilde{A}_\alpha}$ [29], i.e.,

$$R_{\tilde{A}_\alpha} = \alpha/\tilde{A}_\alpha. \tag{B-21}$$

This has been called an "α-plane raised to level α"or a "zSlice" [44]–[46]. Note that $R_{\tilde{A}_\alpha}$ is an IT2 FS all of whose secondary grades equal α (rather than 1 as would be the case for the usual IT2 FS), and that

$$\mathrm{FOU}(R_{\tilde{A}_\alpha}) = \tilde{A}_\alpha. \tag{B-22}$$

Equation (B-22) provides an α-plane with an important interpretation of and connection to an IT2 FS of height α, and $\tilde{A}_0 = \mathrm{FOU}(\tilde{A})$.

Definition B.12: The *horizontal-slice representation* of GT2 FS $\tilde{A}$ is[26]

$$\begin{aligned}\tilde{A} &= \sup_{\alpha\in[0,1]} \alpha \Big/ \left[\int_{x\in X} [a_\alpha(x), b_\alpha(x)]/x\right] \\ &= \sup_{\alpha\in[0,1]} \alpha/\tilde{A}_\alpha = \bigcup_{\alpha\in[0,1]} \alpha/\tilde{A}_\alpha.\end{aligned} \tag{B-23}$$

Definition B.13: A *closed GT2 FS* $\tilde{A}$ is one whose horizontal slices are closed for $\alpha \in [0,1]$.

This paper focuses exclusively on closed GT2 FSs; consequently, when we use "GT2 FS" or "IT2 FS," they are short for "CGT2 FS" or "CIT2 FS," respectively.

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Author's photograph and biography not available at the time of publication.

Supplementary Materials

I. More Examples That Illustrate How to Construct $\alpha-$ Plane FOUs

These examples further illustrate the eight-step procedure for constructing $\alpha-$plane FOUs. They are all for $\alpha' = 0.5$.

Example SM-1. This example illustrates the construction procedure when the LMF and UMF of the left (right) end of the FOUs for $(\tilde{X}_{11})_0$ ($(\tilde{X}_{13})_0$) do not touch; hence, Step 4 of the eight-step procedure is invoked.

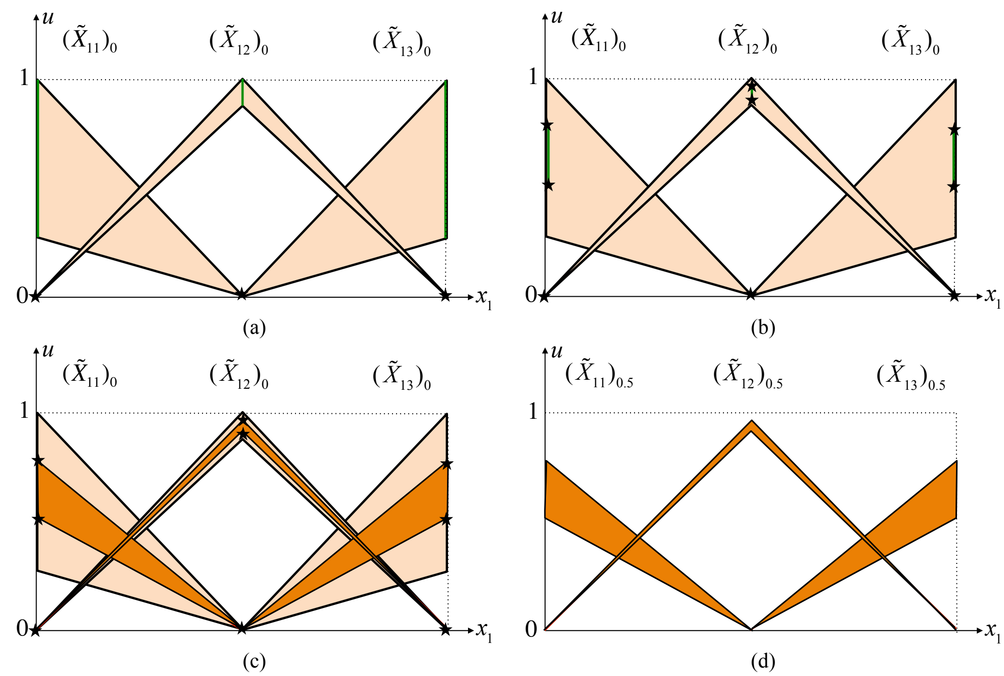


Fig. SM-1. Figures for Example SM-1. (a) FOUs and results of implementing: Steps 1–4, (b) Steps 5 and 6, (c) Step 7, and (d) Step 8.

Example SM-2. The descriptions for this example and the remaining ones in this section are the same as the wordings in the main body's Example 2, and are therefore not repeated here or below.

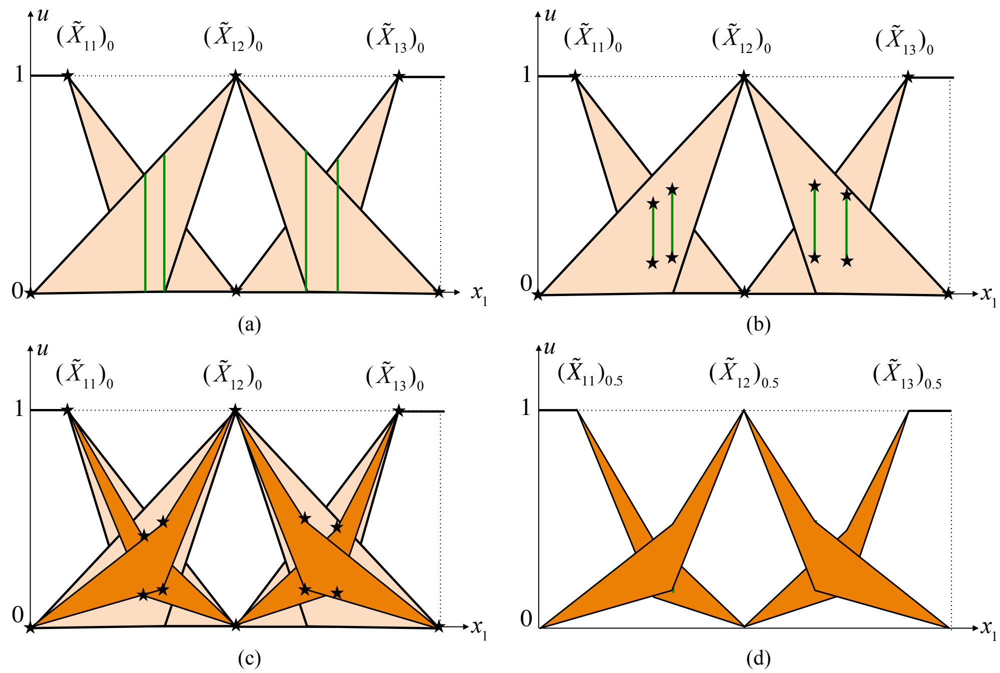


Fig. SM-2. Figures for Example SM-2. (a) FOUs and results of implementing: Steps 1–4, (b) Steps 5 and 6, (c) Step 7, and (d) Step 8.

Example SM-3:

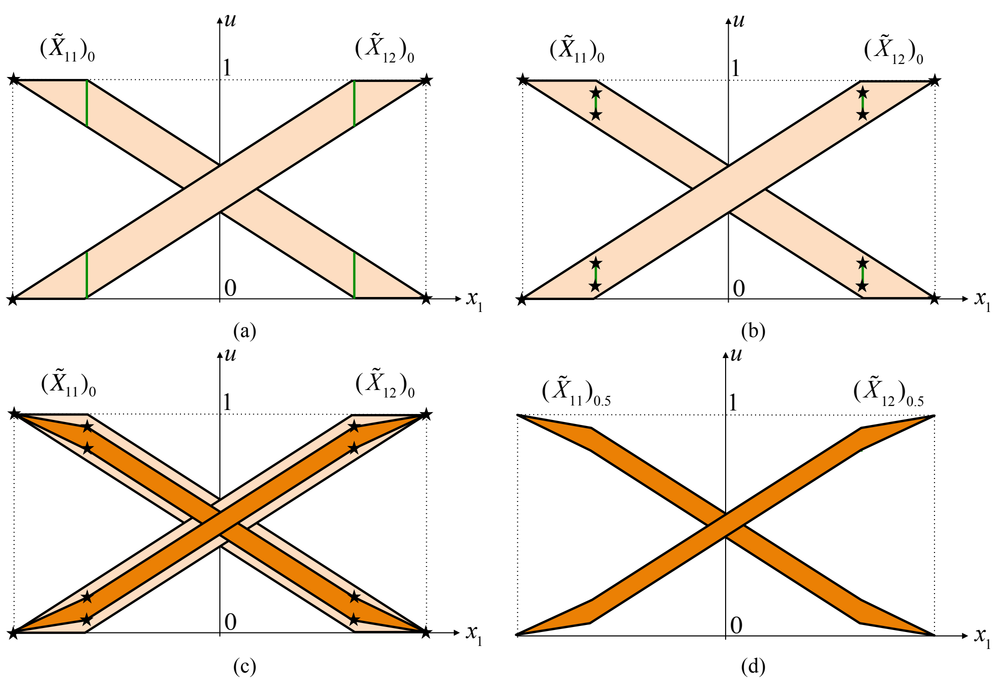


Fig. SM-3. Figures for Example SM-3. (a) FOUs and results of implementing: Steps 1–4, (b) Steps 5 and 6, (c) Step 7, and (d) Step 8.

Example SM-4:

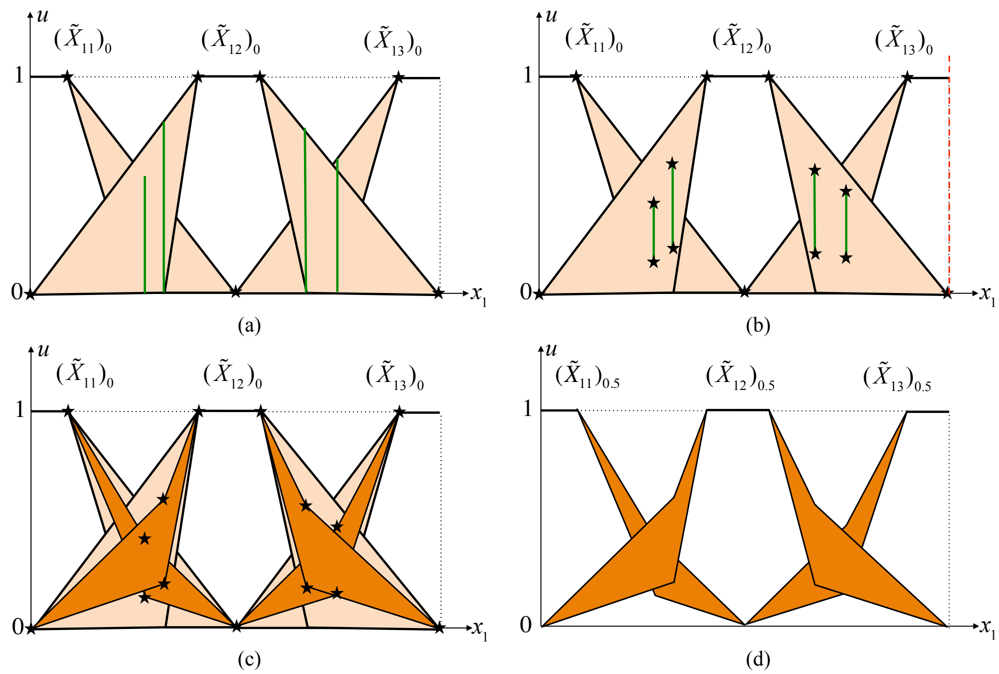


Fig. SM-4. Figures for Example SM-4. (a) FOUs and results of implementing: Steps 1–4, (b) Steps 5 and 6, (c) Step 7, and (d) Step 8.

Example SM-5:

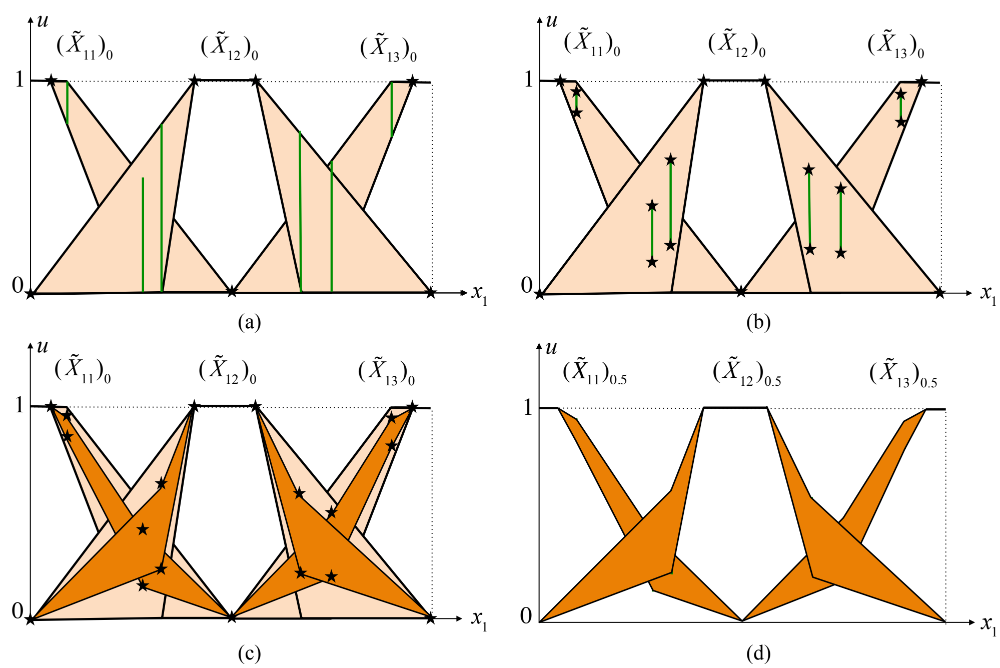


Fig. SM-5. Figures for Example SM-5. (a) FOUs and results of implementing: Steps 1–4, (b) Steps 5 and 6, (c) Step 7, and (d) Step 8.

Example SM-6:

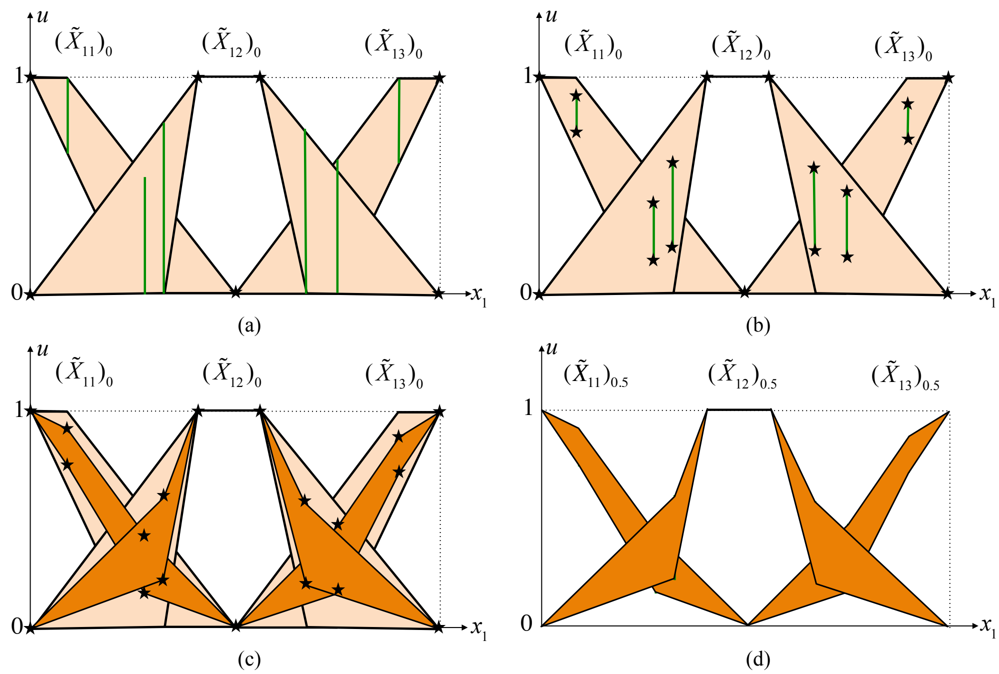


Fig. SM-6. Figures for Example SM-6. (a) FOUs and results of implementing: Steps 1–4, (b) Steps 5 and 6, (c) Step 7, and (d) Step 8.

Example SM-7:

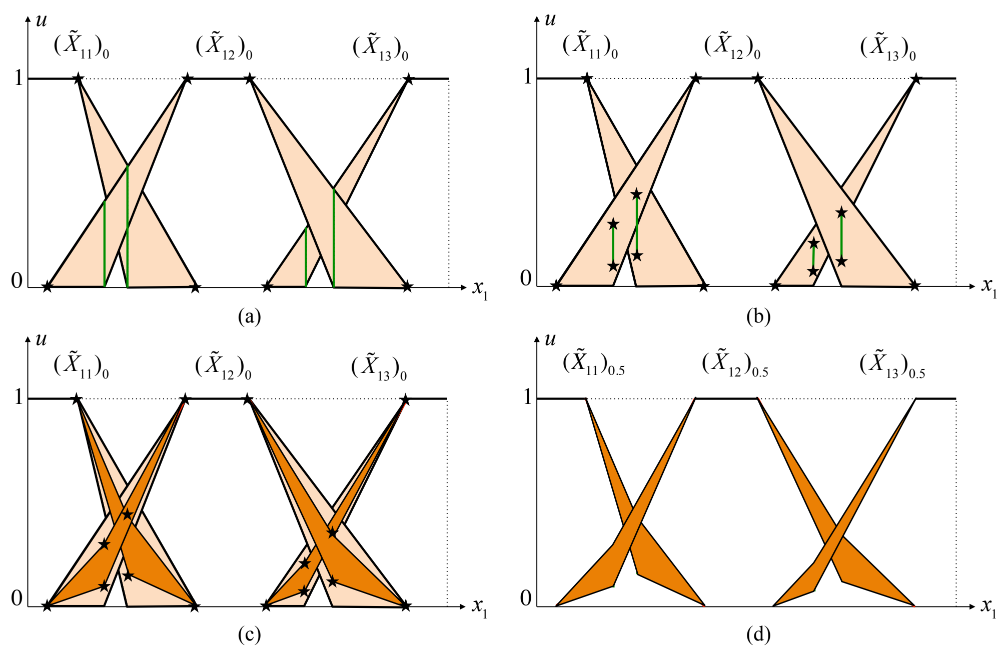


Fig. SM-7. Figures for Example SM-7. (a) FOUs and results of implementing: Steps 1–4, (b) Steps 5 and 6, (c) Step 7, and (d) Step 8.

Example SM-8:

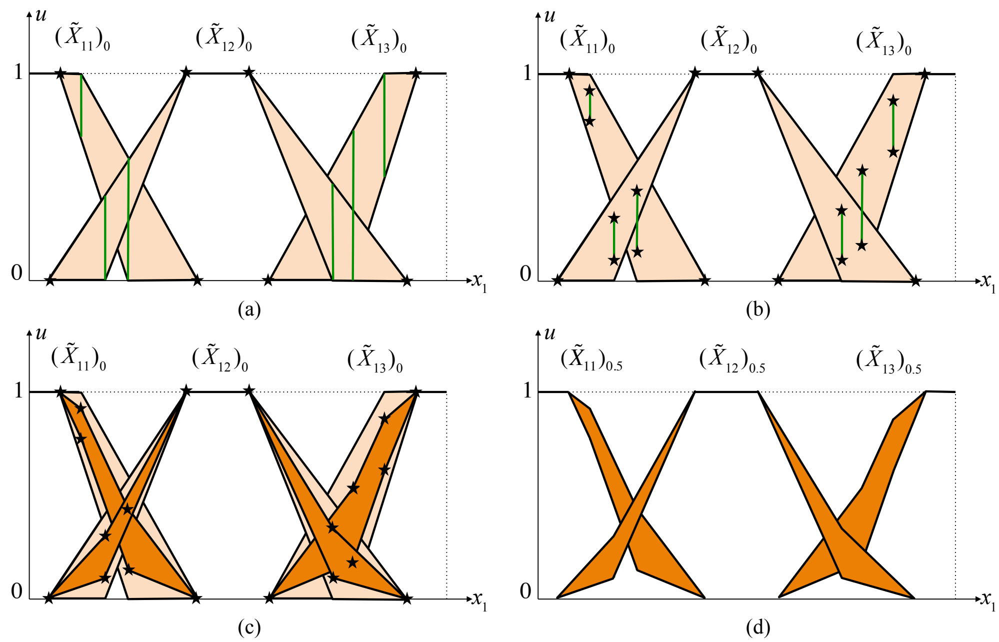


Fig. SM-8. Figures for Example SM-8. (a) FOUs and results of implementing: Steps 1–4, (b) Steps 5 and 6, (c) Step 7, and (d) Step 8.

Example SM-9:

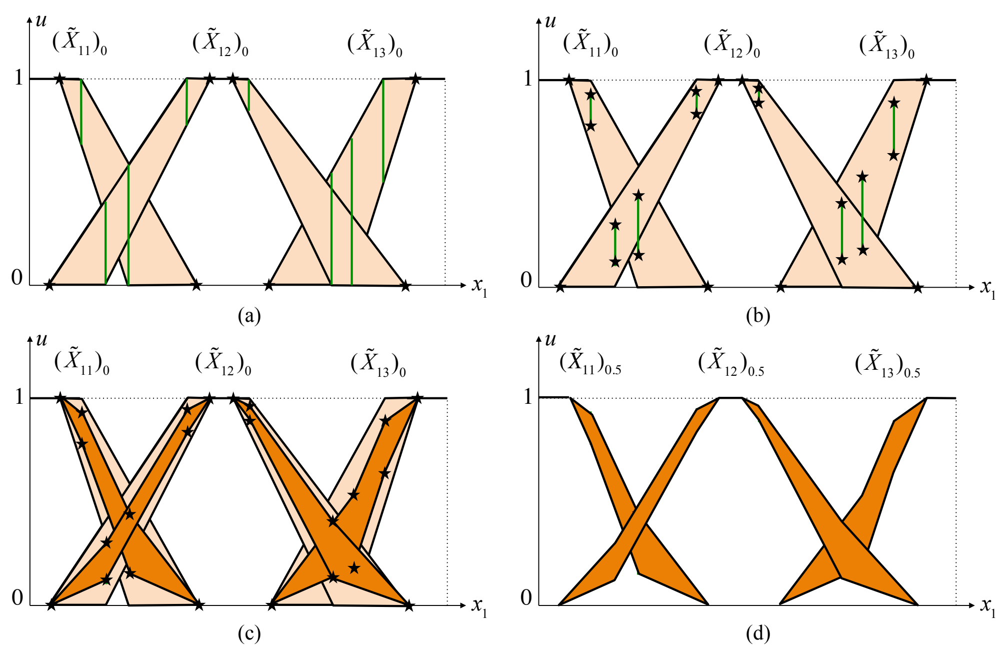


Fig. SM-9. Figures for Example SM-9. (a) FOUs and results of implementing: Steps 1–4, (b) Steps 5 and 6, (c) Step 7, and (d) Step 8.

Example SM-10:

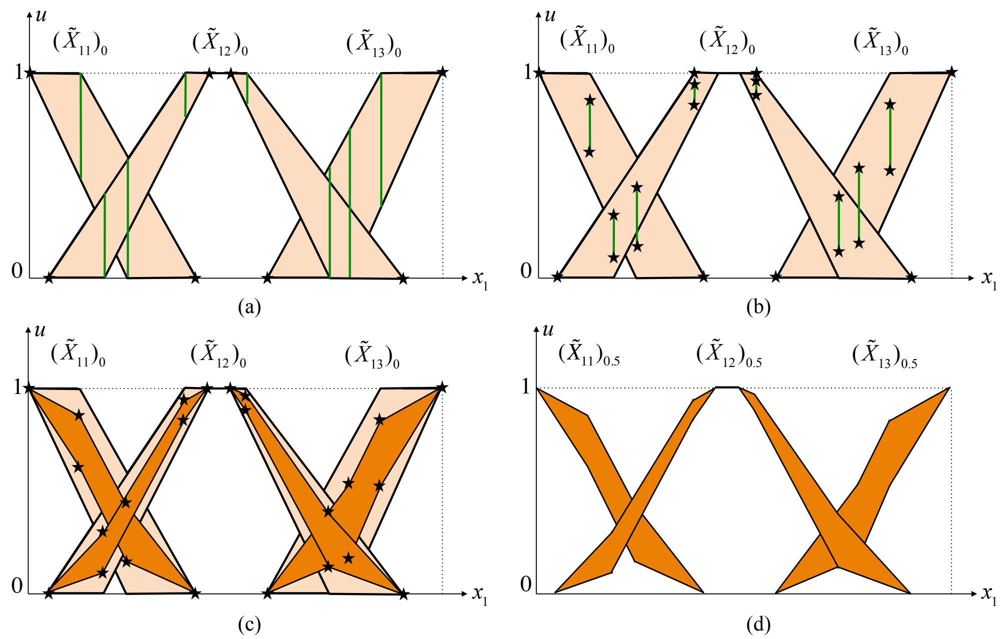


Fig. SM-10. Figures for Example SM-10. (a) FOUs and results of implementing: Steps 1–4, (b) Steps 5 and 6, (c) Step 7, and (d) Step 8.

II. More Examples That Illustrate Partitions for WH GT2 Fuzzy Systems

These examples, are in the same order as, and continue the ones in Section I above, and are analogous to Examples 3 and 4 in the main body of this paper. They further illustrate the Table II two-step procedure for establishing WH GT2(α) first-order rule partitions, and the Table IV four-step procedure for establishing WH GT2(α) second-order rule partitions, but only for $\alpha'=0$ and $\alpha'=0.5$.

Example SM-11: This is our only example for which there are no WH GT2(α) second-order rule partitions. This is due to there being no partition kinks in the two WH GT2(α) first-order rule partitions.

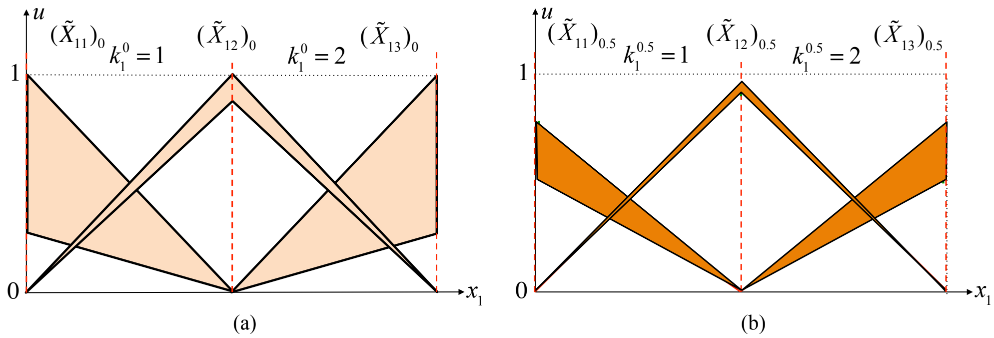


Fig. SM-11. Figures for Example SM-11 WH GT2(α') first- and second-order rule partitions when (a) $\alpha'=0$ and (b) $\alpha'=0.5$. There are no WH GT2(α) second-order rule partitions for this example.

Example SM-12: Observe, in Fig. SM-12, that there are two first-order rule partitions for which $N^2_{GT2(\alpha')}(1|x_1)=N^2_{GT2(\alpha')}(2|x_1)=4$, and $N^2_{GT2(\alpha')}(X_1)=8$ for $\alpha'=0$ and $\alpha'=0.5$. Additionally, observe that the locations and the widths of the WH GT2 second-order rule partitions of X_i at levels $\alpha'=0$ and $\alpha'=0.5$ are the same.

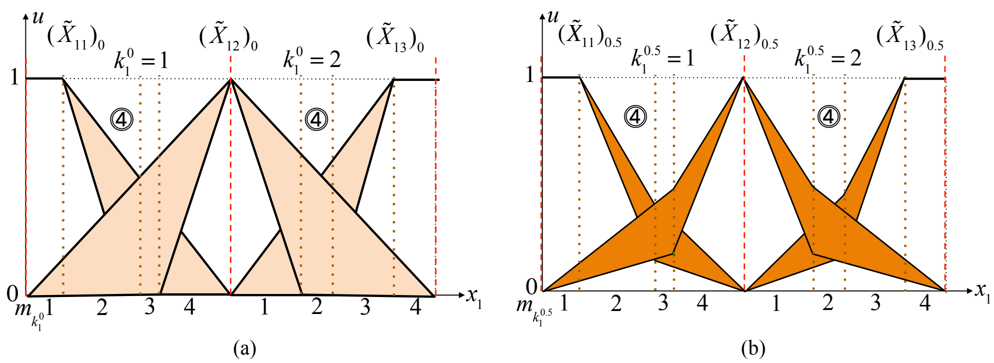
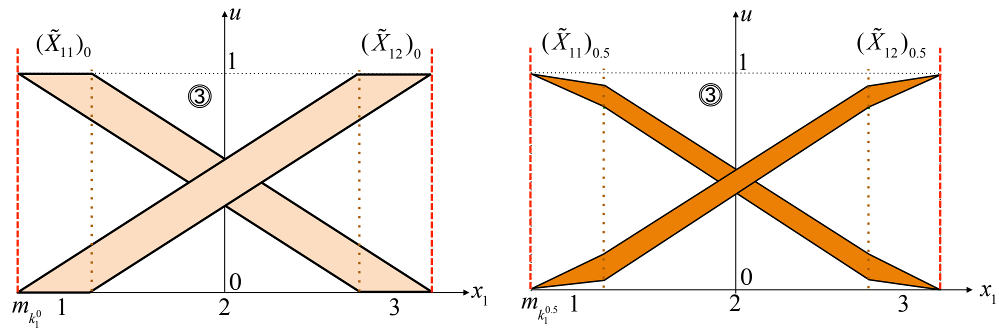


Fig. SM-12. Figures for Example SM-12 WH GT2(α') first- and second-order rule partitions when (a) $\alpha'=0$ and (b) $\alpha'=0.5$. The encircled numbers are $N^2_{GT2(\alpha')}(k_1^{\alpha'}|x_1)$. The numbers along the horizontal axis index the second-order rule partitions within their respective first-order rule partition ($m_{k_1^{\alpha'}}$).

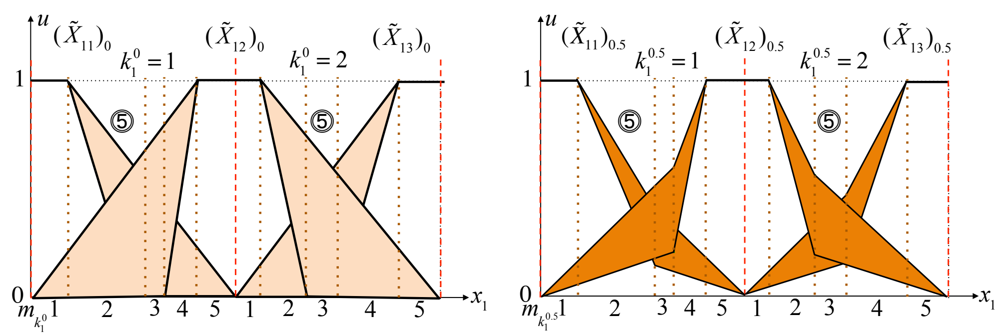
Example SM-13: Observe, in Fig. SM-13, that there is only one first-order rule partition, for which $N^2_{GT2(\alpha')}(1\,|\,x_1)=3$ and $N^2_{GT2(\alpha')}(X_1)=3$ for $\alpha'=0$ and $\alpha'=0.5$. Additionally, observe that the locations and the widths of the WH GT2 second-order rule partitions of X_i at levels $\alpha'=0$ and $\alpha'=0.5$ are the same.



(a) (b)

Fig. SM-13. Figures for Example SM-13 WH GT2 (α') first- and second-order rule partitions when (a) $\alpha'=0$ and (b) $\alpha'=0.5$. The encircled numbers are $N^2_{GT2(\alpha')}(k_1^{\alpha'}\,|\,x_1)$. The numbers along the horizontal axis index the second-order rule partitions within their respective first-order rule partition ($m_{k_1^{\alpha'}}$).

Example SM-14: Observe, in Fig. SM-14, that there are two first-order rule partitions for which $N^2_{GT2(\alpha')}(1\,|\,x_1)=N^2_{GT2(\alpha')}(2\,|\,x_1)=5$, and $N^2_{GT2(\alpha')}(X_1)=10$ for $\alpha'=0$ and $\alpha'=0.5$. Additionally, observe that the locations and the widths of the WH GT2 second-order rule partitions of X_i at levels $\alpha'=0$ and $\alpha'=0.5$ are the same.



(a) (b)

Fig. SM-14. Figures for Example SM-14 WH GT2 (α') first- and second-order rule partitions when (a) $\alpha'=0$ and (b) $\alpha'=0.5$. The encircled numbers are $N^2_{GT2(\alpha')}(k_1^{\alpha'}\,|\,x_1)$. The numbers along the horizontal axis index the second-order rule partitions within their respective first-order rule partition ($m_{k_1^{\alpha'}}$).

Example SM-15: Observe, in Fig. SM-15, that there are two first-order rule partitions for which $N^2_{GT2(\alpha')}(1\,|\,x_1) = N^2_{GT2(\alpha')}(2\,|\,x_1) = 6$, and $N^2_{GT2(\alpha')}(X_1) = 12$ for $\alpha' = 0$ and $\alpha' = 0.5$. Additionally, observe that the locations and the widths of the WH GT2 second-order rule partitions of X_i at levels $\alpha' = 0$ and $\alpha' = 0.5$ are the same.

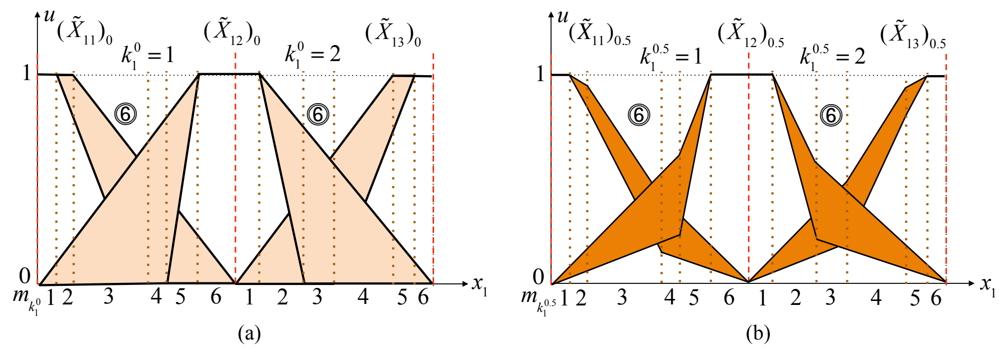


Fig. SM-15. Figures for Example SM-15 WH GT2 (α') first- and second-order rule partitions when (a) $\alpha' = 0$ and (b) $\alpha' = 0.5$. The encircled numbers are $N^2_{GT2(\alpha')}(k_1^{\alpha'}\,|\,x_1)$. The numbers along the horizontal axis index the second-order rule partitions within their respective first-order rule partition ($m_{k_1^{\alpha'}}$).

Example SM-16: Observe, in Fig. SM-16, that there are two first-order rule partitions for which $N^2_{GT2(\alpha')}(1\,|\,x_1) = N^2_{GT2(\alpha')}(2\,|\,x_1) = 5$, and $N^2_{GT2(\alpha')}(X_1) = 10$ for $\alpha' = 0$ and $\alpha' = 0.5$. Additionally, observe that the locations and the widths of the WH GT2 second-order rule partitions of X_i at levels $\alpha' = 0$ and $\alpha' = 0.5$ are the same.

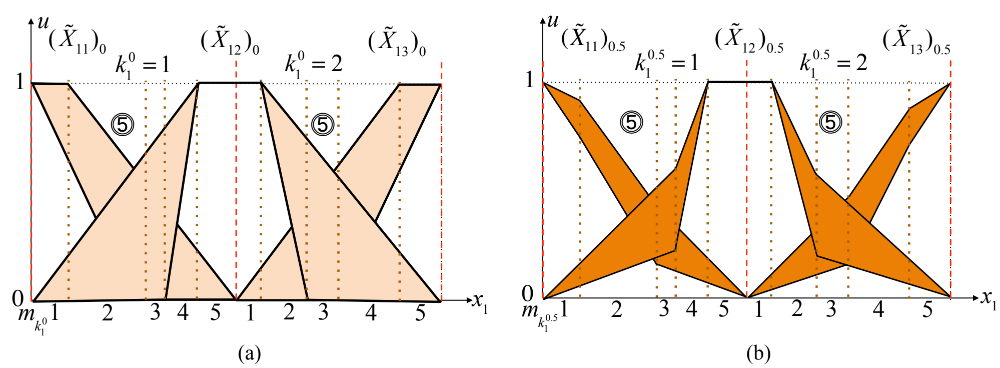


Fig. SM-16. Figures for Example SM-16 WH GT2 (α') first- and second-order rule partitions when (a) $\alpha' = 0$ and (b) $\alpha' = 0.5$. The encircled numbers are $N^2_{GT2(\alpha')}(k_1^{\alpha'}\,|\,x_1)$. The numbers along the horizontal axis index the second-order rule partitions within their respective first-order rule partition ($m_{k_1^{\alpha'}}$).

Example SM-17: Observe, in Fig. SM-17, that there are five first-order rule partitions for which, $N^2_{GT2(\alpha')}(1|x_1) = N^2_{GT2(\alpha')}(5|x_1) = 0$, $N^2_{GT2(\alpha')}(2|x_1) = 5$, $N^2_{GT2(\alpha')}(3|x_1) = 2$ and $N^2_{GT2(\alpha')}(4|x_1) = 3$, and, $N^2_{GT2(\alpha')}(X_1) = 10$ for $\alpha' = 0$ and $\alpha' = 0.5$. Additionally, observe that the locations and the widths of the WH GT2 second-order rule partitions of X_i at levels $\alpha' = 0$ and $\alpha' = 0.5$ are the same.

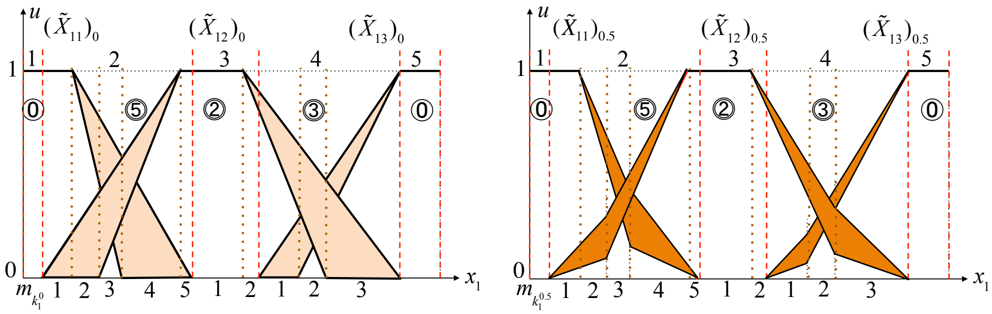


Fig. SM-17. Figures for Example SM-17 WH GT2 (α') first- and second-order rule partitions when (a) $\alpha' = 0$ and (b) $\alpha' = 0.5$. The numbers that appear at the top of each partition (1, 2, …, 5) are $k_1^{\alpha'}$. The encircled numbers are $N^2_{GT2(\alpha')}(k_1^{\alpha'}|x_1)$. The numbers along the horizontal axis index the second-order rule partitions within their respective first-order rule partition ($m_{k_1^{\alpha'}}$).

Example SM-18: Observe, in Fig. SM-18, that there are five first-order rule partitions for which, $N^2_{GT2(\alpha')}(1|x_1) = 0$, $N^2_{GT2(\alpha')}(2|x_1) = 6$, $N^2_{GT2(\alpha')}(3|x_1) = N^2_{GT2(\alpha')}(5|x_1) = 2$ and $N^2_{GT2(\alpha')}(4|x_1) = 4$, and, $N^2_{GT2(\alpha')}(X_1) = 14$ for $\alpha' = 0$ and $\alpha' = 0.5$. Additionally, observe that the locations and the widths of the WH GT2 second-order rule partitions of X_i at levels $\alpha' = 0$ and $\alpha' = 0.5$ are the same.

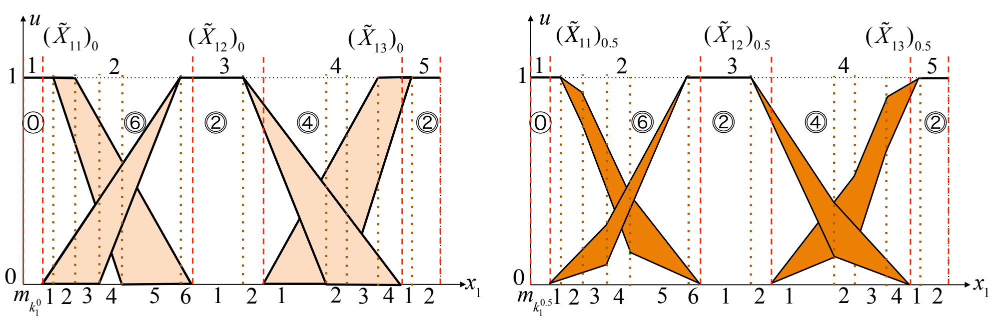


Fig. SM-18. Figures for Example SM-18 WH GT2 (α') first- and second-order rule partitions when (a) $\alpha' = 0$ and (b) $\alpha' = 0.5$. The numbers that appear at the top of each partition (1, 2,…, 5) are $k_1^{\alpha'}$. The encircled numbers are $N^2_{GT2(\alpha')}(k_1^{\alpha'}|x_1)$. The numbers along the horizontal axis index the second-order rule partitions within their respective first-order rule partition ($m_{k_1^{\alpha'}}$).

Example SM-19: Observe, in Fig. SM-19, that there are five first-order rule partitions for which, $N^2_{GT2(\alpha')}(1|x_1)=0$, $N^2_{GT2(\alpha')}(2|x_1)=6$, $N^2_{GT2(\alpha')}(3|x_1)=N^2_{GT2(\alpha')}(4|x_1)=4$ and $N^2_{GT2(\alpha')}(5|x_1)=2$, and, $N^2_{GT2(\alpha')}(X_1)=16$ for $\alpha'=0$ and $\alpha'=0.5$. Additionally, observe that the locations and the widths of the WH GT2 second-order rule partitions of X_i at levels $\alpha'=0$ and $\alpha'=0.5$ are the same.

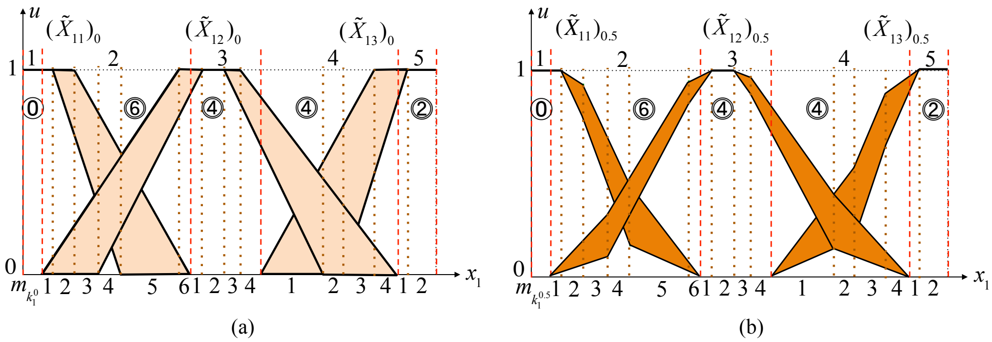


Fig. SM-19. Figures for Example SM-19 WH GT2 (α') first- and second-order rule partitions when (a) $\alpha'=0$ and (b) $\alpha'=0.5$. The numbers that appear at the top of each partition (1, 2,…, 5) are $k_1^{\alpha'}$. The encircled numbers are $N^2_{GT2(\alpha')}(k_1^{\alpha'}|x_1)$. The numbers along the horizontal axis index the second-order rule partitions within their respective first-order rule partition ($m_{k_1^{\alpha'}}$).

Example SM-20: Observe, in Fig. SM-20, that there are five first-order rule partitions for which, $N^2_{GT2(\alpha')}(1|x_1)=N^2_{GT2(\alpha')}(5|x_1)=0$, $N^2_{GT2(\alpha')}(2|x_1)=5$, $N^2_{GT2(\alpha')}(3|x_1)=N^2_{GT2(\alpha')}(4|x_1)=4$ and, $N^2_{GT2(\alpha')}(X_1)=13$ for $\alpha'=0$ and $\alpha'=0.5$. Additionally, observe that the locations and the widths of the WH GT2 second-order rule partitions of X_i at levels $\alpha'=0$ and $\alpha'=0.5$ are the same.

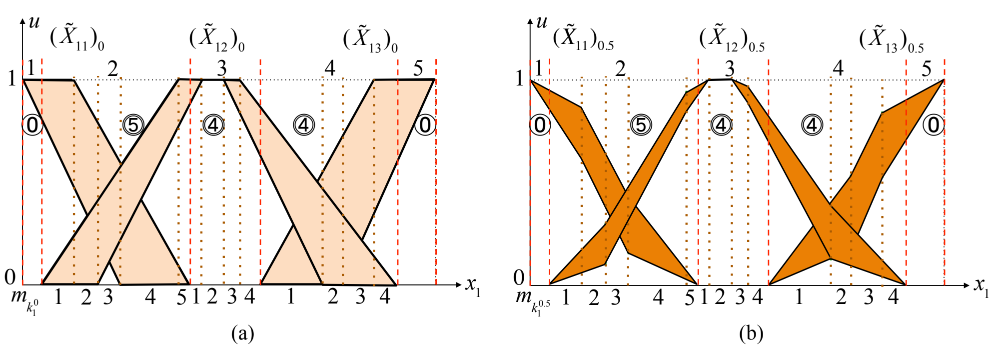


Fig. SM-20. Figures for Example SM-20 WH GT2 (α') first- and second-order rule partitions when (a) $\alpha'=0$ and (b) $\alpha'=0.5$. The numbers that appear at the top of each partition (1, 2,…, 5) are $k_1^{\alpha'}$. The encircled numbers are $N^2_{GT2(\alpha')}(k_1^{\alpha'}|x_1)$. The numbers along the horizontal axis index the second-order rule partitions within their respective first-order rule partition ($m_{k_1^{\alpha'}}$).

III. More Examples That Illustrate Which MFs Change (C) or are Unchanged (UC) in WH GT2 (α') Second-Order Rule Partitions

These examples (which are in the same order as, and continue the ones in Section II above) further illustrate which MFs change or are unchanged in WH GT2 (α') second-order rule partitions when $\alpha' = 0$ and $\alpha' = 0.5$. Each example has a table that is analogous to Table V that summarizes which MFs change (C) or are unchanged (UC) as one moves from one WH GT2 (α) second-order rule partition to the next. For each example, one refers to the appropriate figure that is mentioned in each table, and follows the procedure that is enumerated in Example 5 in the main body of this paper, by sweeping from the left to the right and observing whether or not a LMF or UMF changes or is unchanged as one moves from a second-order partition to the next.

Example SM-21: This is a continuation of Example SM-11. Because there are no WH GT2 (α) second-order rule partitions, there are no changes in a LMF or an UMF that occur within in such partitions. This is a very strange example, and is examined further in Example 6 in the main body of this paper.

Example SM-22: This is a continuation of Example SM-12; so, use Figs. SM-12a and b to obtain the Table SM-1 entries. Examining that table, observe that for $\alpha' = 0$ *8 changes occur*, whereas for $\alpha' = 0.5$ *12 changes occur*, which is a 50% increase in the number of changes. Observe, also, that for $\alpha' = 0$ only *two times* do the changes occur simultaneously for the lower and upper MFs, whereas for $\alpha' = 0.5$, *all of the changes* occur simultaneously (six times) for the lower and upper MFs. The simultaneously occurring pairs are in red.

TABLE SM-1
LMF AND UMF CHANGED (C) OR UNCHANGED (UC) FOR EXAMPLE SM-22

$\alpha' = 0$ (Fig. SM-12a) IT2 fuzzy system: 8C's					
k_1^0	$m_{k_1^0}$	$LMF(\tilde{X}_{11})_0$	$UMF(\tilde{X}_{11})_0$	$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$
1	$1 \to 2$	C	C	UC	UC
1	$2 \to 3$	C	UC	UC	UC
1	$3 \to 4$	UC	UC	C	UC
k_1^0	$m_{k_1^0}$	$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$	$LMF(\tilde{X}_{13})_0$	$UMF(\tilde{X}_{13})_0$
2	$1 \to 2$	C	UC	UC	UC
2	$2 \to 3$	UC	UC	C	UC
2	$3 \to 4$	UC	UC	C	C
$\alpha' = 0.5$ (Fig. SM-12b) WH GT2 fuzzy system: 12C's					
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$LMF(\tilde{X}_{11})_{0.5}$	$UMF(\tilde{X}_{11})_{0.5}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$
1	$1 \to 2$	C	C	UC	UC
1	$2 \to 3$	C	C	UC	UC
1	$3 \to 4$	UC	UC	C	C
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$	$LMF(\tilde{X}_{13})_{0.5}$	$UMF(\tilde{X}_{13})_{0.5}$
2	$1 \to 2$	C	C	UC	UC
2	$2 \to 3$	UC	UC	C	C
2	$3 \to 4$	UC	UC	C	C

Example SM-23: This is a continuation of Example SM-13; so, use Figs. SM-13a and b to obtain the Table SM-2 entries. Examining that table, observe that for $\alpha' = 0$ *4 changes occur*, whereas for $\alpha' = 0.5$ *8 changes occur*, which is a 100% increase in the number of changes. Observe, also, that for $\alpha' = 0$ *no changes* occur simultaneously for the lower and upper MFs, whereas for $\alpha' = 0.5$, *all of the changes* occur simultaneously (four times) for the lower and upper MFs. The simultaneously occurring pairs are in red.

TABLE SM-2
LMF AND UMF CHANGED (C) OR UNCHANGED (UC) FOR EXAMPLE SM-23

$\alpha' = 0$ (Fig. SM-13a) fuzzy system: 4C's					
k_1^0	$m_{k_1^0}$	$LMF(\tilde{X}_{11})_0$	$UMF(\tilde{X}_{11})_0$	$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$
1	$1 \to 2$	UC	C	C	UC
1	$2 \to 3$	C	UC	UC	C
$\alpha' = 0.5$ (Fig. SM-13b) WH GT2 fuzzy system: 8C's					
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$LMF(\tilde{X}_{11})_{0.5}$	$UMF(\tilde{X}_{11})_{0.5}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$
1	$1 \to 2$	C	C	C	C
1	$2 \to 3$	C	C	C	C

Example SM-24: This is a continuation of Example SM-14; so, use Figs. SM-14a and b to obtain the Table SM-3 entries. Examining that table, observe that for $\alpha' = 0.5$ *12 changes occur*, whereas for $\alpha' = 0.5$ *16 changes occur*, which is a 33.33% increase in the number of changes. Observe, also, that for $\alpha' = 0$ only *four times* do the changes occur simultaneously for the lower and upper MFs, whereas for $\alpha' = 0.5$, *all of the changes* occur simultaneously (eight times) for the lower and upper MFs. The simultaneously occurring pairs are in red.

TABLE SM-3
LMF AND UMF CHANGED (C) OR UNCHANGED (UC) FOR EXAMPLE SM-24

$\alpha' = 0$ (Fig. SM-14a) fuzzy system: 12C's					
k_1^0	$m_{k_1^0}$	$LMF(\tilde{X}_{11})_0$	$UMF(\tilde{X}_{11})_0$	$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$
1	$1 \to 2$	C	C	UC	UC
1	$2 \to 3$	C	UC	UC	UC
1	$3 \to 4$	UC	UC	C	UC
1	$4 \to 5$	UC	UC	C	C
k_1^0	$m_{k_1^0}$	$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$	$LMF(\tilde{X}_{13})_0$	$UMF(\tilde{X}_{13})_0$
2	$1 \to 2$	C	C	UC	UC
2	$2 \to 3$	C	UC	UC	UC
2	$3 \to 4$	UC	UC	C	UC
2	$4 \to 5$	UC	UC	C	C
$\alpha' = 0.5$ (Fig. SM-14b) WH GT2 fuzzy system: 16C's					
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$LMF(\tilde{X}_{11})_{0.5}$	$UMF(\tilde{X}_{11})_{0.5}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$
1	$1 \to 2$	C	C	UC	UC
1	$2 \to 3$	C	C	UC	UC
1	$3 \to 4$	UC	UC	C	C
1	$4 \to 5$	UC	UC	C	C
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$	$LMF(\tilde{X}_{13})_{0.5}$	$UMF(\tilde{X}_{13})_{0.5}$
2	$1 \to 2$	C	C	UC	UC
2	$2 \to 3$	C	C	UC	UC
2	$3 \to 4$	UC	UC	C	C
2	$4 \to 5$	UC	UC	C	C

Example SM-25: This is a continuation of Example SM-15; so, use Figs. SM-15a and b to obtain the Table SM-4 entries. Examining that table, observe that for $\alpha' = 0.5$ *12 changes occur*, whereas for $\alpha' = 0.5$ *20 changes occur*, which is a 67.67% increase in the number of changes. Observe, also, that for $\alpha' = 0$ only *two times* do the changes occur simultaneously for the lower and upper MFs, whereas for $\alpha' = 0.5$, *all of the changes* occur simultaneously (10 times) for the lower and upper MFs. The simultaneously occurring pairs are in red.

TABLE SM-4
LMF AND UMF CHANGED (C) OR UNCHANGED (UC) FOR EXAMPLE SM-25

$\alpha' = 0$ (Fig. SM-15a) fuzzy system: 12C's					
k_1^0	$m_{k_1^0}$	$LMF(\tilde{X}_{11})_0$	$UMF(\tilde{X}_{11})_0$	$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$
1	$1 \to 2$	C	UC	UC	UC
1	$2 \to 3$	UC	C	UC	UC
1	$3 \to 4$	C	UC	UC	UC
1	$4 \to 5$	UC	UC	C	UC
1	$5 \to 6$	UC	UC	C	C
k_1^0	$m_{k_1^0}$	$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$	$LMF(\tilde{X}_{13})_0$	$UMF(\tilde{X}_{13})_0$
2	$1 \to 2$	C	C	UC	UC
2	$2 \to 3$	C	UC	UC	UC
2	$3 \to 4$	UC	UC	C	UC
2	$4 \to 5$	UC	UC	UC	C
2	$5 \to 6$	UC	UC	C	UC

$\alpha' = 0.5$ (Fig. SM-15b) WH GT2 fuzzy system: 20C's					
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$LMF(\tilde{X}_{11})_{0.5}$	$UMF(\tilde{X}_{11})_{0.5}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$
1	$1 \to 2$	C	C	UC	UC
1	$2 \to 3$	C	C	UC	UC
1	$3 \to 4$	C	C	UC	UC
1	$4 \to 5$	UC	UC	C	C
1	$5 \to 6$	UC	UC	C	C
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$	$LMF(\tilde{X}_{13})_{0.5}$	$UMF(\tilde{X}_{13})_{0.5}$
2	$1 \to 2$	C	C	UC	UC
2	$2 \to 3$	C	C	UC	UC
2	$3 \to 4$	UC	UC	C	C
2	$4 \to 5$	UC	UC	C	C
2	$5 \to 6$	UC	UC	C	C

Example SM-26: This is a continuation of Example SM-16; so, use Figs. SM-16a and b to obtain the Table SM-5 entries. Examining that table, observe that for $\alpha' = 0.5$ *10 changes occur*, whereas for $\alpha' = 0.5$ *16 changes occur*, which is a 60% increase in the number of changes. Observe, also, that for $\alpha' = 0$ only *two times* do the changes occur simultaneously for the lower and upper MFs, whereas for $\alpha' = 0.5$, *all of the changes* occur simultaneously (eight times) for the lower and upper MFs. The simultaneously occurring pairs are in red.

TABLE SM-5
LMF AND UMF CHANGED (C) OR UNCHANGED (UC) FOR EXAMPLE SM-26

$\alpha' = 0$ (Fig. SM-16a) fuzzy system: 12C's					
k_1^0	$m_{k_1^0}$	$LMF(\tilde{X}_{11})_0$	$UMF(\tilde{X}_{11})_0$	$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$
1	$1 \to 2$	UC	C	UC	UC
1	$2 \to 3$	C	UC	UC	UC
1	$3 \to 4$	UC	UC	C	UC
1	$4 \to 5$	UC	UC	C	C
k_1^0	$m_{k_1^0}$	$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$	$LMF(\tilde{X}_{13})_0$	$UMF(\tilde{X}_{13})_0$
2	$1 \to 2$	C	C	UC	UC
2	$2 \to 3$	C	UC	UC	UC
2	$3 \to 4$	UC	UC	C	UC
2	$4 \to 5$	UC	UC	UC	C
$\alpha' = 0.5$ (Fig. SM-16b) WH GT2 fuzzy system: 16C's					
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$LMF(\tilde{X}_{11})_{0.5}$	$UMF(\tilde{X}_{11})_{0.5}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$
1	$1 \to 2$	C	C	UC	UC
1	$2 \to 3$	C	C	UC	UC
1	$3 \to 4$	UC	UC	C	C
1	$4 \to 5$	UC	UC	C	C
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$	$LMF(\tilde{X}_{13})_{0.5}$	$UMF(\tilde{X}_{13})_{0.5}$
2	$1 \to 2$	C	C	UC	UC
2	$2 \to 3$	C	C	UC	UC
2	$3 \to 4$	UC	UC	C	C
2	$4 \to 5$	UC	UC	C	C

Example SM-27: This is a continuation of Example SM-17; so, use Figs. SM-17a and b to obtain the Table SM-6 entries. Notice, from Figs. SM-17a, b that the first and fifth first-order rule partitions have no second-order rule partitions, which is why $k_1^{\alpha'} = 1$ and $k_1^{\alpha'} = 5$ do not appear in this table. Examining Table SM-6, observe that for $\alpha' = 0.5$ *10 changes occur*, whereas for $\alpha' = 0.5$ *14 changes occur*, which is a 40% increase in the number of changes. Observe, also, that for $\alpha' = 0$ only *3 times* do the changes occur simultaneously for the lower and upper MFs, whereas for $\alpha' = 0.5$, *all of the changes* occur simultaneously (six times) for the lower and upper MFs. The simultaneously occurring pairs are in red.

TABLE SM-6
LMF AND UMF CHANGED (C) OR UNCHANGED (UC) FOR EXAMPLE SM-27

$\alpha' = 0$ (Fig. SM-17a) fuzzy system: 10C's					
k_1^0	$m_{k_1^0}$	$LMF(\tilde{X}_{11})_0$	$UMF(\tilde{X}_{11})_0$	$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$
2	$1 \rightarrow 2$	C	C	UC	UC
2	$2 \rightarrow 3$	UC	UC	C	UC
2	$3 \rightarrow 4$	C	UC	UC	UC
2	$4 \rightarrow 5$	UC	UC	C	C
k_1^0	$m_{k_1^0}$			$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$
3	$1 \rightarrow 2$			C	C
k_1^0	$m_{k_1^0}$	$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$	$LMF(\tilde{X}_{13})_0$	$UMF(\tilde{X}_{13})_0$
4	$1 \rightarrow 2$	UC	UC	C	UC
4	$2 \rightarrow 3$	C	UC	UC	UC
$\alpha' = 0.5$ (Fig. SM-17b) WH GT2 fuzzy system: 14C's					
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$LMF(\tilde{X}_{11})_{0.5}$	$UMF(\tilde{X}_{11})_{0.5}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$
2	$1 \rightarrow 2$	C	C	UC	UC
2	$2 \rightarrow 3$	UC	UC	C	C
2	$3 \rightarrow 4$	C	C	UC	UC
2	$4 \rightarrow 5$	UC	UC	C	C
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$	$LMF(\tilde{X}_{13})_{0.5}$	$UMF(\tilde{X}_{13})_{0.5}$
3	$1 \rightarrow 2$			C	C
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$	$LMF(\tilde{X}_{13})_0$	$UMF(\tilde{X}_{13})_0$
4	$1 \rightarrow 2$	UC	UC	C	C
4	$2 \rightarrow 3$	C	C	UC	UC

Example SM-28: This is a continuation of Example SM-18; so, use Figs. SM-18a and b to obtain the Table SM-7 entries. Notice, from Figs. SM-18a, b that the first first-order rule partitions has no second-order rule partitions, which is why $k_1^{\alpha'} = 1$ does not appear in this table. Examining Table SM-7, observe that for $\alpha' = 0.5$ *12 changes occur*, whereas for $\alpha' = 0.5$ *18 changes occur*, which is a 50% increase in the number of changes. Observe, also, that for $\alpha' = 0$ only *2 times* do the changes occur simultaneously for the lower and upper MFs, whereas for $\alpha' = 0.5$, *all of the changes* occur simultaneously (six times) for the lower and upper MFs. The simultaneously occurring pairs are in red.

TABLE SM-7

LMF AND UMF CHANGED (C) OR UNCHANGED (UC) FOR EXAMPLE SM-28

$\alpha' = 0$ (Fig. SM-18a) fuzzy system: 10C's					
k_1^0	$m_{k_1^0}$	$LMF(\tilde{X}_{11})_0$	$UMF(\tilde{X}_{11})_0$	$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$
2	$1 \to 2$	C	UC	UC	UC
2	$2 \to 3$	UC	C	UC	UC
2	$3 \to 4$	UC	UC	C	UC
2	$4 \to 5$	C	UC	UC	UC
2	$5 \to 6$	UC	UC	C	C
k_1^0	$m_{k_1^0}$			$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$
3	$1 \to 2$			C	C
k_1^0	$m_{k_1^0}$	$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$	$LMF(\tilde{X}_{13})_0$	$UMF(\tilde{X}_{13})_0$
4	$1 \to 2$	C	UC	UC	UC
4	$2 \to 3$	UC	UC	C	UC
4	$3 \to 4$	UC	UC	UC	C
k_1^0	$m_{k_1^0}$			$LMF(\tilde{X}_{13})_0$	$UMF(\tilde{X}_{13})_0$
5	$1 \to 2$			C	UC
$\alpha' = 0.5$ (Fig. SM-18b) WH GT2 fuzzy system: 14C's					
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$LMF(\tilde{X}_{11})_{0.5}$	$UMF(\tilde{X}_{11})_{0.5}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$
2	$1 \to 2$	C	C	UC	UC
2	$2 \to 3$	UC	UC	UC	UC
2	$3 \to 4$	UC	UC	C	C
2	$4 \to 5$	C	C	UC	UC
2	$5 \to 6$	UC	UC	C	C
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$	$LMF(\tilde{X}_{13})_{0.5}$	$UMF(\tilde{X}_{13})_{0.5}$
3	$1 \to 2$			C	C
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$
4	$1 \to 2$	C	C	UC	UC
4	$2 \to 3$	UC	UC	C	C
4	$3 \to 4$	UC	UC	C	C
$k_1^{0.5}$	$m_{k_1^{0.5}}$			$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$
5	$1 \to 2$			C	UC

Example SM-29: This is a continuation of Example SM-19; so, use Figs. SM-19a and b to obtain the Table SM-8 entries. Notice, from Figs. SM-19a, b that the first first-order rule partitions has no second-order rule partitions, which is why $k_1^{\alpha'} = 1$ does not appear in this table. Examining Table SM-8, observe that for $\alpha' = 0.5$ *12 changes occur*, whereas for $\alpha' = 0.5$ *24 changes occur*, which is a 100% increase in the number of changes. Observe, also, that for $\alpha' = 0$ only *at no time* do the changes occur simultaneously for the lower and upper MFs, whereas for $\alpha' = 0.5$, *all of the changes* occur simultaneously (12 times) for the lower and upper MFs. The simultaneously occurring pairs are in red.

TABLE SM-8
LMF AND UMF CHANGED (C) OR UNCHANGED (UC) FOR EXAMPLE SM-29

$\alpha' = 0$ (Fig. SM-19a) fuzzy system: 12C's					
k_1^0	$m_{k_1^0}$	$LMF(\tilde{X}_{11})_0$	$UMF(\tilde{X}_{11})_0$	$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$
2	$1 \to 2$	C	UC	UC	UC
2	$2 \to 3$	UC	C	UC	UC
2	$3 \to 4$	UC	UC	C	UC
2	$4 \to 5$	C	UC	UC	UC
2	$5 \to 6$	UC	UC	UC	C
k_1^0	$m_{k_1^0}$			$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$
3	$1 \to 2$			C	UC
3	$2 \to 3$			C	UC
3	$3 \to 4$			UC	C
k_1^0	$m_{k_1^0}$	$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$	$LMF(\tilde{X}_{13})_0$	$UMF(\tilde{X}_{13})_0$
4	$1 \to 2$	C	UC	UC	UC
4	$2 \to 3$	UC	UC	C	UC
4	$3 \to 4$	UC	UC	UC	C
k_1^0	$m_{k_1^0}$			$LMF(\tilde{X}_{13})_0$	$UMF(\tilde{X}_{13})_0$
5	$1 \to 2$			C	UC

$\alpha' = 0.5$ (Fig. SM-19b) WH GT2 fuzzy system: 24C's					
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$LMF(\tilde{X}_{11})_{0.5}$	$UMF(\tilde{X}_{11})_{0.5}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$
2	$1 \to 2$	C	C	UC	UC
2	$2 \to 3$	C	C	UC	UC
2	$3 \to 4$	UC	UC	C	C
2	$4 \to 5$	C	C	UC	UC
2	$5 \to 6$	UC	UC	C	C
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$	$LMF(\tilde{X}_{13})_{0.5}$	$UMF(\tilde{X}_{13})_{0.5}$
3	$1 \to 2$			C	C
3	$2 \to 3$			C	C
3	$3 \to 4$			C	C
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$
4	$1 \to 2$	C	C	UC	UC
4	$2 \to 3$	UC	UC	C	C
4	$3 \to 4$	UC	UC	C	C
$k_1^{0.5}$	$m_{k_1^{0.5}}$			$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$
5	$1 \to 2$			C	UC

Example SM-30: This is a continuation of Example SM-20; so, use Figs. SM-20a and b to obtain the Table SM-9 entries. Notice, from Figs. SM-20a, b that the first and fifth first-order rule partitions have no second-order rule partitions, which is why $k_1^{\alpha'} = 1$ and $k_1^{\alpha'} = 5$ do not appear in this table. Examining Table SM-9, observe that for $\alpha' = 0.5$ *10 changes occur*, whereas for $\alpha' = 0.5$ *20 changes occur*, which is a 100% increase in the number of changes. Observe, also, that for $\alpha' = 0$ only *at no time* do the changes occur simultaneously for the lower and upper MFs, whereas for $\alpha' = 0.5$, *all of the changes* occur simultaneously (10 times) for the lower and upper MFs. The simultaneously occurring pairs are in red.

TABLE SM-9

LMF AND UMF CHANGED (C) OR UNCHANGED (UC) FOR EXAMPLE SM-30

$\alpha' = 0$ (Fig. SM-20a) fuzzy system: 12C's					
k_1^0	$m_{k_1^0}$	$LMF(\tilde{X}_{11})_0$	$UMF(\tilde{X}_{11})_0$	$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$
2	$1 \to 2$	UC	C	UC	UC
2	$2 \to 3$	UC	UC	C	UC
2	$3 \to 4$	C	UC	UC	UC
2	$4 \to 5$	UC	UC	UC	C
k_1^0	$m_{k_1^0}$			$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$
3	$1 \to 2$			C	UC
3	$2 \to 3$			C	UC
3	$3 \to 4$			UC	C
k_1^0	$m_{k_1^0}$	$LMF(\tilde{X}_{12})_0$	$UMF(\tilde{X}_{12})_0$	$LMF(\tilde{X}_{13})_0$	$UMF(\tilde{X}_{13})_0$
4	$1 \to 2$	C	UC	UC	UC
4	$2 \to 3$	UC	UC	C	UC
4	$3 \to 4$	UC	UC	UC	C
$\alpha' = 0.5$ (Fig. SM-20b) WH GT2 fuzzy system: 24C's					
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$LMF(\tilde{X}_{11})_{0.5}$	$UMF(\tilde{X}_{11})_{0.5}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$
2	$1 \to 2$	C	C	UC	UC
2	$2 \to 3$	UC	UC	C	C
2	$3 \to 4$	C	C	UC	UC
2	$4 \to 5$	UC	UC	C	C
$k_1^{0.5}$	$m_{k_1^{0.5}}$			$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$
3	$1 \to 2$			C	C
3	$2 \to 3$			C	C
3	$3 \to 4$			C	C
$k_1^{0.5}$	$m_{k_1^{0.5}}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$	$LMF(\tilde{X}_{12})_{0.5}$	$UMF(\tilde{X}_{12})_{0.5}$
4	$1 \to 2$	C	C	UC	UC
4	$2 \to 3$	UC	UC	C	C
4	$3 \to 4$	UC	UC	C	C

IV. Data for WH GT2 FPID Controller Novelty Partitions

TABLE SM-10
RANGES, RULES, UMFS, LMFS AND FIRING INTERVALS IN THE FOUR REGIONS (SEE FIG. 8)

	Region 1	Region 2	Region 3	Region 4
Ranges	$E \in [-1,0]$ $\Delta E \in [-1,0]$	$E \in [-1,0]$ $\Delta E \in [0,1]$	$E \in [0,1]$ $\Delta E \in [0,1]$	$E \in [0,1]$ $\Delta E \in [-1,0]$
Rules	$\tilde{R}_Z^1 : (\tilde{N}_E, \tilde{N}_{\Delta E}) \to -1$ $\tilde{R}_Z^2 : (\tilde{N}_E, \tilde{Z}_{\Delta E}) \to -0.5$ $\tilde{R}_Z^4 : (\tilde{Z}_E, \tilde{N}_{\Delta E}) \to -0.5$ $\tilde{R}_Z^5 : (\tilde{Z}_E, \tilde{Z}_{\Delta E}) \to 0$	$\tilde{R}_Z^2 : (\tilde{N}_E, \tilde{Z}_{\Delta E}) \to -0.5$ $\tilde{R}_Z^3 : (\tilde{N}_E, \tilde{P}_{\Delta E}) \to 0$ $\tilde{R}_Z^5 : (\tilde{Z}_E, \tilde{Z}_{\Delta E}) \to 0$ $\tilde{R}_Z^6 : (\tilde{Z}_E, \tilde{P}_{\Delta E}) \to 0.5$	$\tilde{R}_Z^5 : (\tilde{Z}_E, \tilde{Z}_{\Delta E}) \to 0$ $\tilde{R}_Z^6 : (\tilde{Z}_E, \tilde{P}_{\Delta E}) \to 0.5$ $\tilde{R}_Z^8 : (\tilde{P}_E, \tilde{Z}_{\Delta E}) \to 0.5$ $\tilde{R}_Z^9 : (\tilde{P}_E, \tilde{P}_{\Delta E}) \to 1$	$\tilde{R}_Z^4 : (\tilde{Z}_E, \tilde{N}_{\Delta E}) \to -0.5$ $\tilde{R}_Z^5 : (\tilde{Z}_E, \tilde{Z}_{\Delta E}) \to 0$ $\tilde{R}_Z^7 : (\tilde{P}_E, \tilde{N}_{\Delta E}) \to 0$ $\tilde{R}_Z^8 : (\tilde{P}_E, \tilde{Z}_{\Delta E}) \to 0.5$
FOU UMFs	$UMF(\tilde{N}_E)_0 = -E$ $UMF(\tilde{Z}_E)_0 = E+1$ $UMF(\tilde{N}_{\Delta E})_0 = -\Delta E$ $UMF(\tilde{Z}_{\Delta E})_0 = \Delta E + 1$	$UMF(\tilde{N}_E)_0 = -E$ $UMF(\tilde{Z}_E)_0 = E+1$ $UMF(\tilde{Z}_{\Delta E})_0 = 1-\Delta E$ $UMF(\tilde{P}_{\Delta E})_0 = \Delta E$	$UMF(\tilde{Z}_E)_0 = 1-E$ $UMF(\tilde{P}_E)_0 = E$ $UMF(\tilde{Z}_{\Delta E})_0 = 1-\Delta E$ $UMF(\tilde{P}_{\Delta E})_0 = \Delta E$	$UMF(\tilde{Z}_E)_0 = 1-E$ $UMF(\tilde{P}_E)_0 = E$ $UMF(\tilde{N}_{\Delta E})_0 = -\Delta E$ $UMF(\tilde{Z}_{\Delta E})_0 = 1+\Delta E$
FOU LMFs	$LMF(\tilde{N}_E)_0 = -0.2E$ $LMF(\tilde{Z}_E)_0 = 0.9(E+1)$ $LMF(\tilde{N}_{\Delta E})_0 = -0.3\Delta E$ $LMF(\tilde{Z}_{\Delta E})_0 = 0.9(\Delta E+1)$	$LMF(\tilde{N}_E)_0 = -0.2E$ $LMF(\tilde{Z}_E)_0 = 0.9(E+1)$ $LMF(\tilde{Z}_{\Delta E})_0 = 0.9(1-\Delta E)$ $LMF(\tilde{P}_{\Delta E})_0 = 0.3\Delta E$	$LMF(\tilde{Z}_E)_0 = 0.9(1-E)$ $LMF(\tilde{P}_E)_0 = 0.2E$ $LMF(\tilde{Z}_{\Delta E})_0 = 0.9(1-\Delta E)$ $LMF(\tilde{P}_{\Delta E})_0 = 0.3\Delta E$	$LMF(\tilde{Z}_E)_0 = 0.9(1-E)$ $LMF(\tilde{P}_E)_0 = 0.2E$ $LMF(\tilde{N}_{\Delta E})_0 = -0.3\Delta E$ $LMF(\tilde{Z}_{\Delta E})_0 = 0.9(1+\Delta E)$
Horizontal-slice LMF and UMF	$LMF(\tilde{A})_\alpha = LMF(\tilde{A})_0 + \alpha w[UMF(\tilde{A})_\alpha - LMF(\tilde{A})_\alpha]$ $UMF(\tilde{A})_\alpha = UMF(\tilde{A})_0 - \alpha(1-w)[UMF(\tilde{A})_\alpha - LMF(\tilde{A})_\alpha]$ $\tilde{A}$ is a generic GT2 FS			
Firing intervals	Region 1		Region 2	
	$\tilde{R}_Z^1 \begin{cases} \bar{f}_\alpha^1 = UMF(\tilde{N}_E)_\alpha \cdot UMF(\tilde{N}_{\Delta E})_\alpha \\ \underline{f}_\alpha^1 = LMF(\tilde{N}_E)_\alpha \cdot LMF(\tilde{N}_{\Delta E})_\alpha \end{cases}$		$\tilde{R}_Z^2 \begin{cases} \bar{f}_\alpha^2 = UMF(\tilde{N}_E)_\alpha \cdot UMF(\tilde{Z}_{\Delta E})_\alpha \\ \underline{f}_\alpha^2 = LMF(\tilde{N}_E)_\alpha \cdot LMF(\tilde{N}_{\Delta E})_\alpha \end{cases}$	
	$\tilde{R}_Z^2 \begin{cases} \bar{f}_\alpha^2 = UMF(\tilde{N}_E)_\alpha \cdot UMF(\tilde{Z}_{\Delta E})_\alpha \\ \underline{f}_\alpha^2 = LMF(\tilde{N}_E)_\alpha \cdot LMF(\tilde{Z}_{\Delta E})_\alpha \end{cases}$		$\tilde{R}_Z^3 \begin{cases} \bar{f}_\alpha^3 = UMF(\tilde{N}_E)_\alpha \cdot UMF(\tilde{P}_{\Delta E})_\alpha \\ \underline{f}_\alpha^3 = LMF(\tilde{N}_E)_\alpha \cdot LMF(\tilde{P}_{\Delta E})_\alpha \end{cases}$	
	$\tilde{R}_Z^4 \begin{cases} \bar{f}_\alpha^4 = UMF(\tilde{Z}_E)_\alpha \cdot UMF(\tilde{N}_{\Delta E})_\alpha \\ \underline{f}_\alpha^4 = LMF(\tilde{Z}_E)_\alpha \cdot LMF(\tilde{N}_{\Delta E})_\alpha \end{cases}$		$\tilde{R}_Z^5 \begin{cases} \bar{f}_\alpha^5 = UMF(\tilde{Z}_E)_\alpha \cdot UMF(\tilde{Z}_{\Delta E})_\alpha \\ \underline{f}_\alpha^5 = LMF(\tilde{Z}_E)_\alpha \cdot LMF(\tilde{N}_{\Delta E})_\alpha \end{cases}$	
	$\tilde{R}_Z^5 \begin{cases} \bar{f}_\alpha^5 = UMF(\tilde{Z}_E)_\alpha \cdot UMF(\tilde{Z}_{\Delta E})_\alpha \\ \underline{f}_\alpha^5 = LMF(\tilde{Z}_E)_\alpha \cdot LMF(\tilde{Z}_{\Delta E})_\alpha \end{cases}$		$\tilde{R}_Z^6 \begin{cases} \bar{f}_\alpha^6 = UMF(\tilde{Z}_E)_\alpha \cdot UMF(\tilde{P}_{\Delta E})_\alpha \\ \underline{f}_\alpha^6 = LMF(\tilde{Z}_E)_\alpha \cdot LMF(\tilde{P}_{\Delta E})_\alpha \end{cases}$	
	Region 3		Region 4	
	$\tilde{R}_Z^5 \begin{cases} \bar{f}_\alpha^5 = UMF(\tilde{Z}_E)_\alpha \cdot UMF(\tilde{Z}_{\Delta E})_\alpha \\ \underline{f}_\alpha^5 = LMF(\tilde{Z}_E)_\alpha \cdot LMF(\tilde{Z}_{\Delta E})_\alpha \end{cases}$		$\tilde{R}_Z^4 \begin{cases} \bar{f}_\alpha^4 = UMF(\tilde{Z}_E)_\alpha \cdot UMF(\tilde{N}_{\Delta E})_\alpha \\ \underline{f}_\alpha^4 = LMF(\tilde{Z}_E)_\alpha \cdot LMF(\tilde{N}_{\Delta E})_\alpha \end{cases}$	
	$\tilde{R}_Z^6 \begin{cases} \bar{f}_\alpha^6 = UMF(\tilde{Z}_E)_\alpha \cdot UMF(\tilde{P}_{\Delta E})_\alpha \\ \underline{f}_\alpha^6 = LMF(\tilde{Z}_E)_\alpha \cdot LMF(\tilde{P}_{\Delta E})_\alpha \end{cases}$		$\tilde{R}_Z^5 \begin{cases} \bar{f}_\alpha^5 = UMF(\tilde{Z}_E)_\alpha \cdot UMF(\tilde{Z}_{\Delta E})_\alpha \\ \underline{f}_\alpha^5 = LMF(\tilde{Z}_E)_\alpha \cdot LMF(\tilde{Z}_{\Delta E})_\alpha \end{cases}$	
	$\tilde{R}_Z^8 \begin{cases} \bar{f}_\alpha^8 = UMF(\tilde{P}_E)_\alpha \cdot UMF(\tilde{Z}_{\Delta E})_\alpha \\ \underline{f}_\alpha^8 = LMF(\tilde{P}_E)_\alpha \cdot LMF(\tilde{Z}_{\Delta E})_\alpha \end{cases}$		$\tilde{R}_Z^7 \begin{cases} \bar{f}_\alpha^7 = UMF(\tilde{P}_E)_\alpha \cdot UMF(\tilde{N}_{\Delta E})_\alpha \\ \underline{f}_\alpha^7 = LMF(\tilde{P}_E)_\alpha \cdot LMF(\tilde{N}_{\Delta E})_\alpha \end{cases}$	
	$\tilde{R}_Z^9 \begin{cases} \bar{f}_\alpha^9 = UMF(\tilde{P}_E)_\alpha \cdot UMF(\tilde{P}_{\Delta E})_\alpha \\ \underline{f}_\alpha^9 = LMF(\tilde{P}_E)_\alpha \cdot LMF(\tilde{P}_{\Delta E})_\alpha \end{cases}$		$\tilde{R}_Z^8 \begin{cases} \bar{f}_\alpha^8 = UMF(\tilde{P}_E)_\alpha \cdot UMF(\tilde{Z}_{\Delta E})_\alpha) \\ \underline{f}_\alpha^8 = LMF(\tilde{P}_E)_\alpha \cdot LMF(\tilde{Z}_{\Delta E})_\alpha \end{cases}$	

- Ranges can be deduced from Fig. 8.
- Rules can be created from Table VI.
- FOU UMFs and LMFs can be deduced from Fig. 8.

- Formulas for the horizontal-slice LMF and UMF are in (1).
- Formulas for the horizontal-slice firing intervals are for each region's rules, and use (5) when: $p=2$, E is $\tilde{F}_1^l$ and ΔE is $\tilde{F}_2^l$; they also use the product t-norm.

TABLE SM-11

$y_{l,\alpha}^{cos}$ AND $y_{r,\alpha}^{cos}$ ITERATIONS IN THE FOUR REGIONS. IF $y_{l,\alpha}^{(j)}$ $(j \in \{0,1,\ldots,4\})$ IS THE WINNER, THEN $L=j$, AND IF $y_{r,\alpha}^{(k)}$ $(k \in \{0,1,\ldots,4\})$ IS THE WINNER, THEN $R=k$ (SEE FIGS. 9 AND 10)

Region	$y_{l,\alpha}^{cos}$ iterations	$y_{r,\alpha}^{cos}$ iterations
1	$y_{l,\alpha}^{(0)}=\frac{-\underline{f}_\alpha^1-0.5\underline{f}_\alpha^2-0.5\underline{f}_\alpha^4}{\underline{f}_\alpha^1+\underline{f}_\alpha^2+\underline{f}_\alpha^4+\underline{f}_\alpha^5}$ $y_{l,\alpha}^{(1)}=\frac{-\overline{f}_\alpha^1-0.5\underline{f}_\alpha^2-0.5\underline{f}_\alpha^4}{\overline{f}_\alpha^1+\underline{f}_\alpha^2+\underline{f}_\alpha^4+\underline{f}_\alpha^5}$ $y_{l,\alpha}^{(2)}=\frac{-\overline{f}_\alpha^1-0.5\overline{f}_\alpha^2-0.5\underline{f}_\alpha^4}{\overline{f}_\alpha^1+\overline{f}_\alpha^2+\underline{f}_\alpha^4+\underline{f}_\alpha^5}$ $y_{l,\alpha}^{(3)}=\frac{-\overline{f}_\alpha^1-0.5\overline{f}_\alpha^2-0.5\overline{f}_\alpha^4}{\overline{f}_\alpha^1+\overline{f}_\alpha^2+\overline{f}_\alpha^4+\underline{f}_\alpha^5}$ $y_{l,\alpha}^{(4)}=\frac{-\overline{f}_\alpha^1-0.5\overline{f}_\alpha^2-0.5\overline{f}_\alpha^4}{\overline{f}_\alpha^1+\overline{f}_\alpha^2+\overline{f}_\alpha^4+\overline{f}_\alpha^5}$	$y_{r,\alpha}^{(0)}=\frac{-\overline{f}_\alpha^1-0.5\overline{f}_\alpha^2-0.5\overline{f}_\alpha^4}{\overline{f}_\alpha^1+\overline{f}_\alpha^2+\overline{f}_\alpha^4+\overline{f}_\alpha^5}$ $y_{r,\alpha}^{(1)}=\frac{-\underline{f}_\alpha^1-0.5\overline{f}_\alpha^2-0.5\overline{f}_\alpha^4}{\underline{f}_\alpha^1+\overline{f}_\alpha^2+\overline{f}_\alpha^4+\overline{f}_\alpha^5}$ $y_{r,\alpha}^{(2)}=\frac{-\underline{f}_\alpha^1-0.5\underline{f}_\alpha^2-0.5\overline{f}_\alpha^4}{\underline{f}_\alpha^1+\underline{f}_\alpha^2+\overline{f}_\alpha^4+\overline{f}_\alpha^5}$ $y_{r,\alpha}^{(3)}=\frac{-\underline{f}_\alpha^1-0.5\underline{f}_\alpha^2-0.5\underline{f}_\alpha^4}{\underline{f}_\alpha^1+\underline{f}_\alpha^2+\underline{f}_\alpha^4+\overline{f}_\alpha^5}$ $y_{r,\alpha}^{(4)}=\frac{-\underline{f}_\alpha^1-0.5\underline{f}_\alpha^2-0.5\underline{f}_\alpha^4}{\underline{f}_\alpha^1+\underline{f}_\alpha^2+\underline{f}_\alpha^4+\underline{f}_\alpha^5}$
2	$y_{l,\alpha}^{(0)}=\frac{-0.5\underline{f}_\alpha^2+0.5\underline{f}_\alpha^6}{\underline{f}_\alpha^2+\underline{f}_\alpha^3+\underline{f}_\alpha^5+\underline{f}_\alpha^6}$ $y_{l,\alpha}^{(1)}=\frac{-0.5\overline{f}_\alpha^2+0.5\underline{f}_\alpha^6}{\overline{f}_\alpha^2+\underline{f}_\alpha^3+\underline{f}_\alpha^5+\underline{f}_\alpha^6}$ $y_{l,\alpha}^{(2)}=\frac{-0.5\overline{f}_\alpha^2+0.5\underline{f}_\alpha^6}{\overline{f}_\alpha^2+\overline{f}_\alpha^3+\underline{f}_\alpha^5+\underline{f}_\alpha^6}$ $y_{l,\alpha}^{(3)}=\frac{-0.5\overline{f}_\alpha^2+0.5\underline{f}_\alpha^6}{\overline{f}_\alpha^2+\overline{f}_\alpha^3+\overline{f}_\alpha^5+\underline{f}_\alpha^6}$ $y_{l,\alpha}^{(4)}=\frac{-0.5\overline{f}_\alpha^2+0.5\overline{f}_\alpha^6}{\overline{f}_\alpha^2+\overline{f}_\alpha^3+\overline{f}_\alpha^5+\overline{f}_\alpha^6}$	$y_{r,\alpha}^{(0)}=\frac{-0.5\overline{f}_\alpha^2+0.5\overline{f}_\alpha^6}{\overline{f}_\alpha^2+\overline{f}_\alpha^3+\overline{f}_\alpha^5+\overline{f}_\alpha^6}$ $y_{r,\alpha}^{(1)}=\frac{-0.5\underline{f}_\alpha^2+0.5\overline{f}_\alpha^6}{\underline{f}_\alpha^2+\overline{f}_\alpha^3+\overline{f}_\alpha^5+\overline{f}_\alpha^6}$ $y_{r,\alpha}^{(2)}=\frac{-0.5\underline{f}_\alpha^2+0.5\overline{f}_\alpha^6}{\underline{f}_\alpha^2+\underline{f}_\alpha^3+\overline{f}_\alpha^5+\overline{f}_\alpha^6}$ $y_{r,\alpha}^{(3)}=\frac{-0.5\underline{f}_\alpha^2+0.5\overline{f}_\alpha^6}{\underline{f}_\alpha^2+\underline{f}_\alpha^3+\underline{f}_\alpha^5+\overline{f}_\alpha^6}$ $y_{r,\alpha}^{(4)}=\frac{-0.5\underline{f}_\alpha^2+0.5\underline{f}_\alpha^6}{\underline{f}_\alpha^2+\underline{f}_\alpha^3+\underline{f}_\alpha^5+\underline{f}_\alpha^6}$
3	$y_{l,\alpha}^{(0)}=\frac{0.5\underline{f}_\alpha^6+0.5\underline{f}_\alpha^8+\underline{f}_\alpha^9}{\underline{f}_\alpha^5+\underline{f}_\alpha^6+\underline{f}_\alpha^8+\underline{f}_\alpha^9}$ $y_{l,\alpha}^{(1)}=\frac{0.5\underline{f}_\alpha^6+0.5\underline{f}_\alpha^8+\underline{f}_\alpha^9}{\overline{f}_\alpha^5+\underline{f}_\alpha^6+\underline{f}_\alpha^8+\underline{f}_\alpha^9}$ $y_{l,\alpha}^{(2)}=\frac{0.5\overline{f}_\alpha^6+0.5\underline{f}_\alpha^8+\underline{f}_\alpha^9}{\overline{f}_\alpha^5+\overline{f}_\alpha^6+\underline{f}_\alpha^8+\underline{f}_\alpha^9}$ $y_{l,\alpha}^{(3)}=\frac{0.5\overline{f}_\alpha^6+0.5\overline{f}_\alpha^8+\underline{f}_\alpha^9}{\overline{f}_\alpha^5+\overline{f}_\alpha^6+\overline{f}_\alpha^8+\underline{f}_\alpha^9}$ $y_{l,\alpha}^{(4)}=\frac{0.5\overline{f}_\alpha^6+0.5\overline{f}_\alpha^8+\overline{f}_\alpha^9}{\overline{f}_\alpha^5+\overline{f}_\alpha^6+\overline{f}_\alpha^8+\overline{f}_\alpha^9}$	$y_{r,\alpha}^{(0)}=\frac{0.5\overline{f}_\alpha^6+0.5\overline{f}_\alpha^8+\overline{f}_\alpha^9}{\overline{f}_\alpha^5+\overline{f}_\alpha^6+\overline{f}_\alpha^8+\overline{f}_\alpha^9}$ $y_{r,\alpha}^{(1)}=\frac{0.5\overline{f}_\alpha^6+0.5\overline{f}_\alpha^8+\overline{f}_\alpha^9}{\underline{f}_\alpha^5+\overline{f}_\alpha^6+\overline{f}_\alpha^8+\overline{f}_\alpha^9}$ $y_{r,\alpha}^{(2)}=\frac{0.5\underline{f}_\alpha^6+0.5\overline{f}_\alpha^8+\overline{f}_\alpha^9}{\underline{f}_\alpha^5+\underline{f}_\alpha^6+\overline{f}_\alpha^8+\overline{f}_\alpha^9}$ $y_{r,\alpha}^{(3)}=\frac{0.5\underline{f}_\alpha^6+0.5\underline{f}_\alpha^8+\overline{f}_\alpha^9}{\underline{f}_\alpha^5+\underline{f}_\alpha^6+\underline{f}_\alpha^8+\overline{f}_\alpha^9}$ $y_{r,\alpha}^{(4)}=\frac{0.5\underline{f}_\alpha^6+0.5\underline{f}_\alpha^8+\underline{f}_\alpha^9}{\underline{f}_\alpha^5+\underline{f}_\alpha^6+\underline{f}_\alpha^8+\underline{f}_\alpha^9}$
4	$y_{l,\alpha}^{(0)}=\frac{-0.5\underline{f}_\alpha^4+0.5\underline{f}_\alpha^8}{\underline{f}_\alpha^4+\underline{f}_\alpha^5+\underline{f}_\alpha^7+\underline{f}_\alpha^8}$ $y_{l,\alpha}^{(1)}=\frac{-0.5\overline{f}_\alpha^4+0.5\underline{f}_\alpha^8}{\overline{f}_\alpha^4+\underline{f}_\alpha^5+\underline{f}_\alpha^7+\underline{f}_\alpha^8}$ $y_{l,\alpha}^{(2)}=\frac{-0.5\overline{f}_\alpha^4+0.5\underline{f}_\alpha^8}{\overline{f}_\alpha^4+\overline{f}_\alpha^5+\underline{f}_\alpha^7+\underline{f}_\alpha^8}$ $y_{l,\alpha}^{(3)}=\frac{-0.5\overline{f}_\alpha^4+0.5\underline{f}_\alpha^8}{\overline{f}_\alpha^4+\overline{f}_\alpha^5+\overline{f}_\alpha^7+\underline{f}_\alpha^8}$ $y_{l,\alpha}^{(4)}=\frac{-0.5\overline{f}_\alpha^4+0.5\overline{f}_\alpha^8}{\overline{f}_\alpha^4+\overline{f}_\alpha^5+\overline{f}_\alpha^7+\overline{f}_\alpha^8}$	$y_{r,\alpha}^{(0)}=\frac{-0.5\overline{f}_\alpha^4+0.5\overline{f}_\alpha^8}{\overline{f}_\alpha^4+\overline{f}_\alpha^5+\overline{f}_\alpha^7+\overline{f}_\alpha^8}$ $y_{r,\alpha}^{(1)}=\frac{-0.5\underline{f}_\alpha^4+0.5\overline{f}_\alpha^8}{\underline{f}_\alpha^4+\overline{f}_\alpha^5+\overline{f}_\alpha^7+\overline{f}_\alpha^8}$ $y_{r,\alpha}^{(2)}=\frac{-0.5\underline{f}_\alpha^4+0.5\overline{f}_\alpha^8}{\underline{f}_\alpha^4+\underline{f}_\alpha^5+\overline{f}_\alpha^7+\overline{f}_\alpha^8}$ $y_{r,\alpha}^{(3)}=\frac{-0.5\underline{f}_\alpha^4+0.5\overline{f}_\alpha^8}{\underline{f}_\alpha^4+\underline{f}_\alpha^5+\underline{f}_\alpha^7+\overline{f}_\alpha^8}$ $y_{r,\alpha}^{(4)}=\frac{-0.5\underline{f}_\alpha^4+0.5\underline{f}_\alpha^8}{\underline{f}_\alpha^4+\underline{f}_\alpha^5+\underline{f}_\alpha^7+\underline{f}_\alpha^8}$

To obtain the formulas in this table:

- $y_{l,\alpha}^{(j)}$: Use (6), the numbering of the fired rules in each region (from Table SM-10), and the crisp rule consequents (see "Rules" in Table SM-10), which play the role of the $c_l(\tilde{G}_\alpha^i)$ in (6).
- $y_{r,\alpha}^{(k)}$: Use (7), the numbering of the fired rules in each region (from Table SM-10), and the crisp rule consequents (see "Rules" in Table SM-10), which play the role of the $c_r(\tilde{G}_\alpha^i)$ in (7). Note that because the rule consequents are crisp, $c_r(\tilde{G}_\alpha^i)=c_l(\tilde{G}_\alpha^i)$.