Explaining the Performance Potential of Rule-Based Fuzzy Systems as a *Greater Sculpting of the State Space*

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***Abstract*—This paper provides some new and novel application-independent perspectives on why improved performance usually occurs as one goes from crisp, to type-1 (T1), and to interval type-2 (IT2) fuzzy systems, by introducing three kinds of partitions: (1) *Uncertainty partitions* that let us distinguish T1 fuzzy sets from crisp sets, and IT2 fuzzy sets from T1 fuzzy sets; (2) *First-and second-order rule partitions* that are direct results of uncertainty partitions, and are associated with the number of rules that fire in different regions of the state space, and, the changes in their mathematical formulae within those regions; and (3) *Novelty partitions* that can only occur in an IT2 fuzzy system that uses type-reduction. Rule and novelty partitions sculpt the state space into hyperrectangles within each of which resides a different nonlinear function. It is the author's conjecture that the greater sculpting of the state space by a T1 fuzzy system lets it outperform a crisp system, and the even greater sculpting of the state space by an IT2 fuzzy system lets it outperform a T1 fuzzy system. The latter can occur even when the T1 and IT2 fuzzy systems are described by the same number of parameters.**

***Index Terms*—Interval type-2 (IT2) fuzzy system, novelty partitions, rule-based fuzzy systems, rule partitions, type-1 (T1) fuzzy system, uncertainty partitions.**

I. Introduction

SINCE the seminal work of [57], a very important application for fuzzy sets has been *rule-based fuzzy systems*. When such systems use type-1 (T1) [interval type-2 (IT2) or general type-2 (GT2)] fuzzy sets they are called *T1* [*IT2* or *GT2*] *fuzzy systems*. Thousands of articles (including books) have been published about such fuzzy systems,[1] and invariably they demonstrate that better performance[2] [as measured by an application's performance metric(s)] is achieved by:

1) a T1 fuzzy system over a nonfuzzy system;
2) an IT2 fuzzy system over a T1 fuzzy system; and
3) a GT2 fuzzy system over an IT2 fuzzy system.

A crucial question is: *Why does improved performance occur as one goes from crisp, to T1, to IT2, to GT2 fuzzy systems?*

Some possible earlier answers to this question are (as paraphrased by this author):

1) A fuzzy system is inherently nonlinear (NL) and so it should perform better than a linear system, because most real-world systems are NL. But, there can be many kinds of NL systems and so what is so special about the NL nature of a fuzzy system? Assessing nonlinearity is not an easy task, but there have been attempts at performing it for estimators and stochastic systems [31], [28], cryptographic functions such as hash functions, block ciphers, and stream ciphers [6], and open-loop dynamical systems [46], [19]. There is also a study to assess the nonlinearity of fuzzy (PID) controllers that compared them to their nonfuzzy counterparts [20]. Nevertheless, the nature of nonlinearity of fuzzy systems has to be further studied to answer the above question.
2) A fuzzy system easily achieves smooth transitions across its (input) state space because of its overlapping membership functions (MFs) [53], i.e., its variable structure, something that cannot be achieved (easily or at all) using a nonfuzzy system. But, how does one quantify this? Again, there have been studies to demonstrate the benefit of smooth NL approximations [14] over linear approximations, and on the smoothness of approximations that vary in form in subdomains of the input space [12] but serious studies are needed to quantify this for fuzzy systems.
3) A T1 fuzzy system usually has more design degrees of freedom than does a nonfuzzy system, because of the parameters that are needed to define T1 MFs. An IT2 (GT2) fuzzy system has more design degrees of freedom than does a T1 (IT2) fuzzy system because it takes more parameters to define an IT2 (GT2) fuzzy set than it does to define a T1 (IT2) fuzzy set. Some recent studies [7], [44], [8] indicate that it is not the number of MF parameters that lead to improved performance; rather, there is something inherently different about the ways that an IT2 (GT2) fuzzy system handles MF uncertainties that leads to this.

Two earlier works that provide some interesting and new insights about why improved performance may occur for an IT2

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This paper has supplementary downloadable material available at http://ieeexplore.ieee.org, provided by the author. This includes rule partition examples, Data for IT2 FPID Controller Novelty Partitions and Control Surfaces for IT2 FPID Controllers Designs. This material is 3.21 MB in size.

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

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[1]Most are about T1 fuzzy systems, followed by IT2 fuzzy systems, and more recently there are some on GT2 fuzzy systems. Examples of such publications are: [1], [9]–[11], [13], [16]–[18], [26], [33] (this contains 83 reprinted papers), [34], [37]–[39], and [56].

[2]Interpretability, as a performance metric, is outside of the scope of this paper.

fuzzy system over a T1 fuzzy system are [55] and [36]. The former is based on observations from a fuzzy logic control application and uses the novel concept of equivalent T1 fuzzy sets, whereas the latter is based on one kind of an IT2 fuzzy system and uses an expansion of that system about a T1 fuzzy system. A key thought from the latter work is: "... a traditional comparative performance analysis begins with a specific application and is performed entirely within the context of that application, with each application requiring its own performance analysis. [The author's] contention is that there should be a common component to all performance analyses, after which the rest of the performance analysis is application-dependent."

This paper provides new and novel answers to the above crucial question that are in the spirit of "a common component to all performance analyses," but only for T1 and IT2 fuzzy systems, leaving the extension of its results for GT2 fuzzy systems as future research. In a nutshell, it will be shown that a T1 fuzzy system can sculpt its state space with greater variability than a crisp rule-based system can, and in ways that cannot be accomplished by the crisp system, and that an IT2 fuzzy system (that has the same number of rules as the T1 fuzzy system) can sculpt the state space with even greater variability, and in ways that cannot be accomplished by a T1 fuzzy system. This approach may provide tools for characterizing and mathematically quantifying the nature of the "excess nonlinearity" provided by IT2 fuzzy systems in the design process. It will lead to the following conjecture: *it is the greater sculpting of the state space that lets an IT2 fuzzy system usually outperform a T1 fuzzy system, and a T1 fuzzy system usually outperform a crisp system.*

II. Background

A. Uncertainty Partitions

In a rule-based system, each antecedent is a term that is associated with a linguistic variable. If each antecedent, x_i, $(i = 1, \dots, p)$ is described by Q_i linguistic terms $T_{x_i} = \{X_{ij}\}_{j=1}^{Q_i}$, then it may be said that these terms *partition* x_i. Partitions come in different guises.[3] Four different kinds of partitions of $x_i = x$ are depicted in Fig. 1. The horizontal axis of each figure is x (e.g., temperature, price of stock on a particular day, blood pressure, etc.) and the vertical axis of this figure is the degree of belonging of each x in a partition, also called the membership of x in a partition.

Definition 1: A *crisp partition* [as in Fig. 1(a)] of the real variable x comprises nonoverlapping adjacent regions that are intervals of real numbers, where the degree of membership in each region is 1. They can be described mathematically using classical (crisp) sets.

In Fig. 1(a), each of the intervals, e.g., $[0, a]$, $(a, b]$, etc., is a *crisp partition*, and a given value of x can only reside in one of them with full membership of 1. Additionally, each crisp partition is associated with a linguistic term, e.g., VL or L, etc.,

[3]The partitions in this paper are not the pioneering *fuzzy partitions* introduced in [43], which have received considerable attention in the literature, e.g., [2], [5], [21].

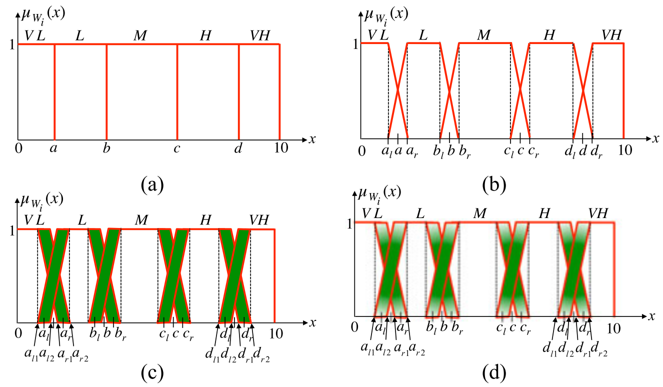


Fig. 1. Four kinds of partitions: (a) crisp, (b) first-order uncertainty, (c) second-order uncertainty with uniform weighting, and (d) second-order uncertainty with nonuniform weighting. W_i denotes the ith word, where $W_1 = VL$, $W_2 = L, W_3 = M, W_4 = H$, and $W_5 = VH$.

and there is always a sharp transition from one term to the next at $x = a, b, c$, or d.

Crisp partitions serve us well in some situations, but they do not permit any uncertainty about a, b, c, or d. Such uncertainty can be expressed by letting all numbers about which there is uncertainty become an interval of numbers, e.g., $a \to [a_l, a_r]$, $b \to [b_l, b_r]$, etc.

Definition 2: A *first-order uncertainty partition* [as in Fig. 1(b)] of the real variable, x, comprises overlapping intervals, where one is absolutely certain about where the overlap begins and ends, so that the degree of membership in each region of overlap is a real number that is an element of [0,1]. They can be described mathematically using classical (T1) fuzzy sets.

First-order uncertainty partitions lead to smooth transitions from one region (linguistic term) to another, which is very different from the sharp transitions that occur when crisp partitions are used. Such partitions serve us well in many situations, but they do not allow for any uncertainty about the interval end-points, which can once again be expressed by letting all numbers about which there is uncertainty become an interval of numbers, e.g., $a \to [[a_{l1}, a_{l2}], [a_{r1}, a_{r2}]]$, etc.

Definition 3: A *second-order uncertainty partition* [as in Fig. 1(c)] of the real-variable x comprises overlapping intervals where one is unsure about where the overlap begins and ends, so that the degree of membership in each region of overlap is an interval of real numbers that is a subset of $[0, 1]$.

Definition 4: Each region in $X \times [0, 1]$, in which the degree of membership is an interval of real numbers, is called the *footprint of uncertainty* (FOU) of W_i [35], [37].

Definition 5: A uniformly (nonuniformly) shaded FOU [as in Fig. 1(c) and (d)] denotes a uniform (nonuniform) weighting of all of its points, and is called a *uniformly* (*nonuniformly*) *weighted second-order uncertainty partition*. Uniformly (nonuniformly) weighted partitions can be described mathematically using IT2 (GT2) fuzzy sets.

Readers are no doubt already anticipating additional levels of uncertainty, along the lines just given, e.g., $a \to [[[a_{l1_l}, a_{l1_r}], [a_{l2_l}, a_{l2_r}]], [[a_{r1_l}, a_{r1_r}], [a_{r2_l}, a_{r2_r}]]]$; however, $[[[a_{l1_l}, a_{l1_r}], [a_{l2_l}, a_{l2_r}]], [[a_{r1_l}, a_{r1_r}], [a_{r2_l}, a_{r2_r}]]]$ is equivalent to

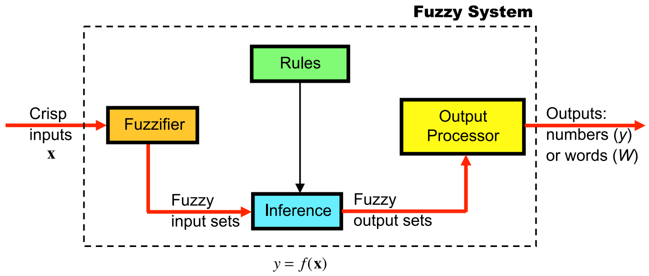


Fig. 2. Fuzzy system.

$[[a_{l1_l}, a_{l2_r}], [a_{r1_l}, a_{r2_r}]]$ so, *a second-order uncertainty model suffices.*[4]

B. Rule-Based Fuzzy Systems

A *rule-based fuzzy system* (*fuzzy system*, for short) contains four components—*rules*, *fuzzifier*, *inference (engine)*, and *output processor*—that are interconnected as shown in Fig. 2 (e.g., [32], [37], [22], [23]). Once the rules have been established, the fuzzy system can be viewed as a mapping from inputs to outputs, and this mapping can be expressed quantitatively as $y = f(\mathbf{x})$. Brief discussions are given next only for those parts of this system that are needed in the rest of this paper.

C. Rules

Suppose that a fuzzy system has p inputs $x_1 \in X_1, \ldots, x_p \in X_p$, and one output $y \in Y$, where x_i is described by Q_i linguistic terms $T_{x_i} = \{X_{ij}\}_{j=1}^{Q_i}$, and y is either described by Q_y linguistic terms, $T_y = \{Y_j\}_{j=1}^{Q_y}$, or by a function $g(x_1, \ldots, x_p)$.

Definition 6: The structures of the *antecedents* of an lth generic *Zadeh rule* (R_Z^l) [57] and *TSK rule* (R_{TSK}^l) [47], [45] for a fuzzy system are the same, namely $(l = 1, \ldots, M)$ "x_1 is F_1^l and ... and x_p is F_p^l" (where $F_i^l \in T_{x_i}$); but, the *consequent* of R_Z^l is "y is G^l" and the *consequent* of R_{TSK}^l is "y is $g^l(x_1, \ldots, x_p)$." In a T1 (IT2) fuzzy system, F_j^l and G^l are T1 fuzzy sets ($F_j^l \to \tilde{F}_j^l$ and $G^l \to \tilde{G}^l$, where $\tilde{F}_j^l$ and $\tilde{G}^l$ are IT2 fuzzy sets). The rules $R_Z^l(R_{\mathrm{TSK}}^l)$ are used in a *Mamdani (TSK) fuzzy system* [32] ([45], [47]).

Because $G^l \in T_y$ is a fuzzy set, it is described by its MF $\mu_{G^l}(y)$. In R_{TSK}^l, although y does not seem to be a fuzzy set, it can be modeled as a *fuzzy singleton* G^l, so R_{TSK}^l can be made to resemble R_Z^l, e.g., for T1 fuzzy sets, $\mu_{G^l}(y) \equiv 1$ when $y = g^l(\mathbf{x})$ and $\mu_{G^l}(y) \equiv 0$ otherwise. In this way it is possible to unify fuzzy systems that use either R_Z^l or R_{TSK}^l.

D. Firing Level (Interval) in a T1 (IT2) Fuzzy System

For a T1 (IT2) fuzzy system, T1 (IT2) fuzzy logic principles are used to map T1 (IT2) fuzzy input sets in $X_1 \times \cdots \times X_p$, that flow through a set of IF-THEN rules, into a T1 (IT2) fuzzy output set in Y. The focus of this paper is primarily on the interaction of each T1 (IT2) input with its respective T1 (IT2) antecedent, which then collectively leads to a *firing level* (*interval*) that is the same for both Mamdani and TSK T1 (IT2) fuzzy systems. To keep things as simple as possible, we assume *singleton fuzzification*, although the approach that is taken herein is conceptually the same regardless of the nature of the fuzzifier.

It is well known that, for T1 (IT2) singleton fuzzification, when $\mathbf{x} = \mathbf{x}'$ the firing level (interval) $f^l(\mathbf{x}')$ ($[\underline{f}^l(\mathbf{x}'), \bar{f}^l(\mathbf{x}')]$) for each T1 (IT2) rule $(l = 1, \ldots, M)$ is (e.g., [29], [35], [37])

$$\begin{cases} f^l(\mathbf{x}') = T_{i=1}^p \mu_{F_i^l}(x'_i) \\ [\underline{f}^l(\mathbf{x}'), \bar{f}^l(\mathbf{x}')] = [T_{i=1}^p \underline{\mu}_{\tilde{F}_i^l}(x'_i), T_{i=1}^p \bar{\mu}_{\tilde{F}_i^l}(x'_i)] \end{cases}. \quad (1)$$

In (1), T denotes a t-norm, usually the minimum or product, and $\underline{\mu}_{\tilde{F}_i^l}$ and $\bar{\mu}_{\tilde{F}_i^l}$ denote the lower and upper MFs (LMFs and UMFs) of $\tilde{F}_i^l$, each of which is a T1 fuzzy set that bounds the FOU of the IT2 fuzzy set. Observe that in (1), for a T1 fuzzy system $\mathbf{x}'$ is processed once nonlinearly, but for an IT2 fuzzy system it is processed twice nonlinearly, once using LMFs and once using UMFs.

Definition 7: In a T1 (IT2) fuzzy system, a firing level (interval) is said to *contribute* to its output only if it is nonzero. In a T1 fuzzy system, it is the T1 MFs of rule antecedents that establish exactly where this occurs in $X_1 \times X_2 \times \cdots \times X_p$, and it occurs when the T1 MFs of *all* antecedents are simultaneously nonzero. In an IT2 fuzzy system, it is the UMFs of rule antecedent FOUs that establish exactly where this occurs in $X_1 \times X_2 \times \cdots \times X_p$, and it occurs when the UMF of *all* antecedent FOUs are simultaneously nonzero.[5]

E. Type-Reduction (TR) and Defuzzification

In an IT2 fuzzy system, after the firing intervals have been computed there can be different ways to use them to obtain its final output (e.g., [24], [40], [4]). This paper focuses on doing this by using center-of sets TR (COS TR) followed by defuzzification, because COS TR is arguably the most widely used TR. However, the results in this paper are also applicable to height TR and centroid TR.

Definition 8: COS TR in an IT2 fuzzy system maps mixtures of the lower and upper values of the firing intervals into y_l^{COS} and y_r^{COS}, i.e., in an IT2 Mamdani fuzzy system (similar equations occur for normalized A2-C0 and A2-C1 IT2 TSK fuzzy systems [37, Secs. 9.6.4.2 and 9.6.4.4])

$$y_l^{\mathrm{COS}}(\mathbf{x}') = \frac{\sum_{i=1}^{L} c_l(\tilde{G}^i)\bar{f}^i(\mathbf{x}') + \sum_{i=L+1}^{M} c_l(\tilde{G}^i)\underline{f}^i(\mathbf{x}')}{\sum_{i=1}^{L} \bar{f}^i(\mathbf{x}') + \sum_{i=L+1}^{M} \underline{f}^i(\mathbf{x}')} \quad (2)$$

$$y_r^{\mathrm{COS}}(\mathbf{x}') = \frac{\sum_{i=1}^{R} c_r(\tilde{G}^i)\underline{f}^i(\mathbf{x}') + \sum_{i=R+1}^{M} c_r(\tilde{G}^i)\bar{f}^i(\mathbf{x}')}{\sum_{i=1}^{R} \underline{f}^i(\mathbf{x}') + \sum_{i=R+1}^{M} \bar{f}^i(\mathbf{x}')}. \quad (3)$$

In (2) and (3), $c_l(\tilde{G}^i)$ and $c_r(\tilde{G}^i)$ are the left and right endpoints of the centroid of the IT2 rule-consequent $\tilde{G}^i$, and the

[4]It is conceivable that uncertainty about the filling of the FOU could lead to higher than second-order uncertainty about the FOU, but this is beyond the scope of this paper.

[5]If the UMF is zero then the LMF must also be zero because a LMF can never be larger than the UMF.

switch points L and R have to be computed iteratively, which can be done by using many different published algorithms, the most widely used being KM [25], EKM [52], and EIASC [54]. The *defuzzified output* of the IT2 Mamdani fuzzy system, $y_{\text{COS}}(\mathbf{x}')$, is computed as

$$y_{\text{COS}}(\mathbf{x}') = [y_l^{\text{COS}}(\mathbf{x}') + y_r^{\text{COS}}(\mathbf{x}')]/2. \tag{4}$$

Regarding the nonlinear nature[6] of this IT2 fuzzy system, when (2) and (3) are substituted into (4), one finds that $\underline{f}^i(\mathbf{x}')$ and $\bar{f}^i(\mathbf{x}')$ are combined *quadratically*, including self- and cross-product terms. In a T1 fuzzy system that uses COS defuzzification, firing levels are only combined *linearly*.

III. Rule Partitions

This section defines and illustrates first-order and second-order rule partitions because they help us to understand what is happening in a fuzzy system, T1 or IT2.

A. First-Order Rule Partitions in a Fuzzy System

When a firing level (interval) is computed using either the minimum or product t-norms, then its nonzero occurrence over $X_1 \times X_2 \times \cdots \times X_p$ (see Definition 7) can be established by examining the components of the firing level (interval) separately over each X_i and then combining those results for all $i = 1, \ldots, p$, by using either the minimum or product t-norms, because

$$\begin{cases} \min\ (\text{any } f_i = 0, \text{all other } f_i) = 0 \\ \text{product } (\text{any } f_i = 0, \text{all other } f_i) = 0 \end{cases}. \tag{5}$$

These facts justify defining a first-order rule partition initially for each X_i and then for $X_1 \times X_2$ and $X_1 \times X_2 \times \cdots \times X_p$.

1) First-Order Rule Partitions for Each X_i:

Definition 9: In a T1 (IT2) fuzzy system, a *T1 (IT2) first-order rule partition*[7] of X_i is a collection of nonoverlapping intervals in X_i, in each of which the *same* number of *same* rules is fired whose firing levels (intervals) contribute to the output of that system.

Notations for T1 (IT2) first-order rule partitions are given in Table I. When discussions below are for T1 fuzzy systems, $*$ is replaced by "T1", and when they are for IT2 fuzzy systems, $*$ is replaced by "IT2."

A formal *two-step procedure* for establishing $P_*^1(k_{x_i}|x_i)$, $N_R(k_{x_i})$, and $N_*^1(X_i)$ on a plot (sketch) of the MFs (FOUs) of x_i, is given in Table II.

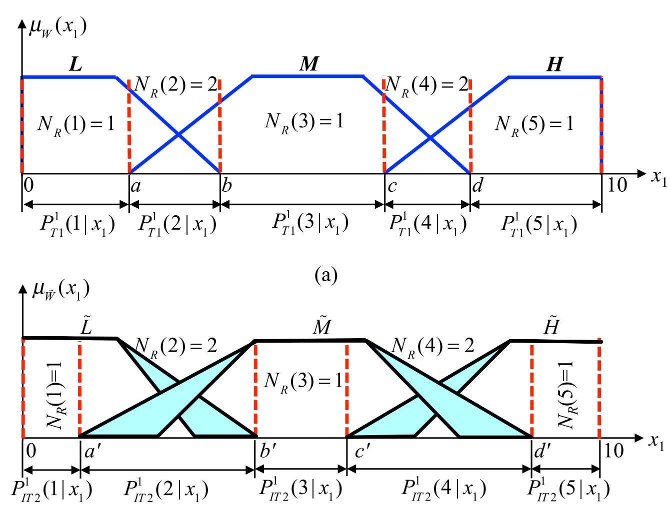
Example 1: Consider $x_1 \in [0, 10]$ covered by three T1 FSs, as depicted in Fig. 3(a), for which there are three T1 (Zadeh or TSK) rules whose antecedents are: R^1 : IF x_1 is L, R^2 : IF x_1 is M, and, R^3 : IF x_1 is H. The results for Step 1 in Table II are shown on Fig. 3(a). Clearly, $N_{\text{T1}}^1(X_1) = 5$. Next, all of the

TABLE I
NOTATIONS USED FOR FIRST-ORDER RULE PARTITIONS. IN THIS TABLE, SUBSCRIPT $*$ REFERS TO EITHER T1 OR IT2

First-Order Rule Partitions	
Symbol	Definition $(i = 1, \ldots, p)$
$P_*^1(k_{x_i}\|x_i)$	T1 or IT2 first-order rule partition of X_i
k_{x_i}	Counter/index of T1 or IT2 first-order rule partition of X_i; $k_{x_i} = 1, \ldots, N_*^1(X_i)$
$N_*^1(X_i)$	Total number of T1 or IT2 first-order rule partitions of X_i
$N_R(k_{x_i})$	Fixed number of same rules fired in each $P_*^1(k_{x_i}\|x_i)$
$P_*^1(k_{x_1}, k_{x_2}, \ldots, k_{x_p})$	T1 or IT2 first-order rule partition of $X_1 \times X_2 \times \cdots \times X_p$, numbered $(k_{x_1}, k_{x_2}, \ldots, k_{x_p})$
$N_*^1(X_1, X_2, \ldots, X_p)$	Total number of T1 or IT2 first-order rule partitions of $X_1 \times X_2 \times \cdots \times X_p$ [use (6)]
$N_R(k_{x_1}, k_{x_2}, \ldots, k_{x_p})$	Fixed number of rules that are fired in each $P_*^1(k_{x_1}, k_{x_2}, \ldots, k_{x_p})$ [use (7)]

TABLE II
TWO-STEP PROCEDURE FOR ESTABLISHING T1 (IT2) FIRST-ORDER RULE PARTITION QUANTITIES FOR A SINGLE VARIABLE, x_i, IN A T1 (IT2) FUZZY SYSTEM, ON A PLOT (SKETCH) OF ITS MFS (FOUS)

Step	Description
1	Scan the axis of x_i with an imaginary dashed vertical line from left to right. Count the number of intersections of this line with the MFs (FOUs) of x_i; they represent the number of *same* rules $[N_R(k_{x_i})]$ whose firing levels (intervals) contribute to the output of a T1 (IT2) fuzzy system. When this number, or the nature of the same rules, changes draw a dashed vertical line; it represents the boundary of a T1 (IT2) first-order rule partition. Insert a dashed vertical line at the start and at the end of X_i. For each x_i, the interval of real numbers between adjacent dashed vertical lines is its T1 (IT2) first-order rule partition.
2	Count the number of $P_*^1(k_{x_i}\|x_i)$, the total being $N_*^1(X_i)$; then, $k_{x_i} = 1, \ldots, N_*^1(X_i)$.



(b)

Fig. 3. Example 1 figures: (a) T1 fuzzy sets and (b) IT2 fuzzy sets.

[6]See [37, Table 9.5] for descriptions of the nonlinear natures of three other kinds of IT2 fuzzy systems.

[7][49] calls these "partitions," but does not distinguish between first- and second-order partitions, something that only occurred to the author during the writing of [37].

TABLE III
FOUR-STEP PROCEDURE FOR ESTABLISHING T1 (IT2) FIRST-ORDER RULE PARTITION QUANTITIES FOR $X_1 \times X_2$ IN A T1 (IT2) FUZZY SYSTEM

Step	Description
1	Locate the T1 (IT2) first-order rule partitions of $x_1 (x_2)$ on the horizontal (vertical) axis, and establish $N_R(k_{x_1})$, $N_R(k_{x_2})$, $N_*^1(X_1)$, and $N_*^1(X_2)$
2	Extend all dashed T1 (IT2) first-order rule partitions (turning them into solid lines) so that they cover $X_1 \times X_2$. The results from doing this will be a collection of rectangles (or squares).
3	Compute $N_R(k_{x_1}, k_{x_2})$ using (7).
4	Compute $N_*^1(X_1, X_2)$ using (6).

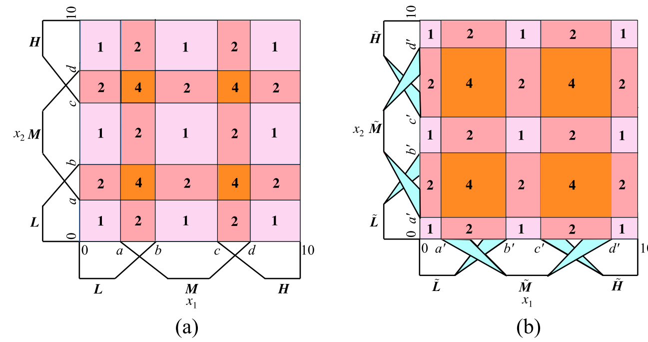


Fig. 4. Example 2 figures for Table III's Steps 1–3: (a) T1 fuzzy system, and (b) IT2 fuzzy system.

T1 fuzzy sets (FSs) in the three rules and in Fig. 3(a) are replaced by the IT2 FSs in[8] Fig. 3(b). The results for Step 1 in Table II are now shown in Fig. 3(b). Clearly, $N_{\text{IT2}}^1(X_1) = 5$. In this example, the number of T1 (IT2) first-order rule partitions is greater than the number of MFs that cover X_1, but this may not always be true (see Example 5). What is noticeably different about the five IT2 first-order rule partitions in Fig. 3(b) is: the one-rule IT2 first-order rule partitions are shorter in length than their T1 counterparts in Fig. 3(a), and, the two-rule IT2 first-order rule partitions are longer in length than their T1 counterparts. This can be summarized, as: *It seems that MF uncertainty tends to fire more rules more often* (at least in this example).

2) First-Order Rule Partitions for $X_1 \times X_2$:

Definition 10: In a T1 (IT2) fuzzy system, a *T1 (IT2) first-order rule partition* of[9] $X_1 \times X_2$ is a collection of non-overlapping rectangles (or squares) in $X_1 \times X_2$, in each of which the *same* number of *same* rules is fired whose firing levels (intervals) contribute to the output of a T1 (IT2) fuzzy system.

Notations for T1 (IT2) first-order rule partitions of $X_1 \times X_2$ are in the bottom half of Table I (set $p = 2$). On a plot (sketch) of the MFs of x_1 on the horizontal axis and the MFs of x_2 on the vertical axis, a formal *four-step procedure* for establishing $P_*^1(k_{x_1}, k_{x_2})$, $N_R(k_{x_1}, k_{x_2})$, and $N_*^1(X_1, X_2)$ is given in Table III. In order to implement this procedure, one must first complete the Table II two-step procedure for establishing $(i = 1, 2)$ $P_*^1(k_{x_i}|x_i)$, $N_R(k_{x_i})$ and $N_*^1(X_i)$.

Example 2: This is an extension of Example 1 from one to two variables, in which $x_1, x_2 \in [0, 10]$ and both variables are covered by the three MFs (FOUs) that are depicted in Fig. 3(a) and (b), for which there are nine (Zadeh or TSK) rules whose antecedents are: R^1 (R^2, R^3) : IF x_1 is L and x_2 is L (M, H), R^4 (R^5, R^6) : IF x_1 is M and x_2 is L (M, H), and R^7 (R^8, R^9) : IF x_1 is H and x_2 is L (M, H). The results for Steps 1–3 in Table III are shown in Fig. 4(a) and (b). In order to keep these figures as uncluttered as possible, we have left off the detailed notations that are in Fig. 3. Note that $k_{x_1}, k_{x_2} = 1, \ldots, 5$ begin in the lowest left-hand square and sweep upward lexicographically from left to right. The numbers that appear in the different colored rectangles (squares) denote $N_R(k_{x_1}, k_{x_2})$; the "1" regions have one fired rule and occur where $N_R(k_{x_1}) = 1$ and $N_R(k_{x_2}) = 1$ [e.g., when $(k_{x_1}, k_{x_2}) = (3, 3)$ rule R^5 fires]; the "2" regions have two fired rules and occur either where $N_R(k_{x_1}) = 1$ and $N_R(k_{x_2}) = 2$ or where $N_R(k_{x_1}) = 2$ and $N_R(k_{x_2}) = 1$ [e.g., when $(k_{x_1}, k_{x_2}) = (4, 3)$ rules R^5 and R^8 fire]; and, the "4" regions have four fired rules and occur where $N_R(k_{x_1}) = 2$ and $N_R(k_{x_2}) = 2$ [e.g., when $(k_{x_1}, k_{x_2}) = (2, 4)$ rules R^2, R^3, R^5, and R^6 fire].

Counting the total number of T1 and IT2 first-order rule partitions on Fig. 4(a) and (b), respectively, one obtains $N_{T1}^1(X_1, X_2) = N_{IT2}^1(X_1, X_2) = 25$. In summary, there are 25 T1 and IT2 first-order rule partitions for this nine-rule T1 (IT2) fuzzy system. So, the T1 (IT2) fuzzy system with only 9 rules and 25 first-order rule partitions is able to provide a much greater sculpting of $X_1 \times X_2$ than is a 9-rule crisp rule based system that can only partition $X_1 \times X_2$ into 9 regions, one for each rule.

Comparing Fig. 4(a) and (b), one observes that one-rule IT2 first-order rule partitions are much smaller in size than their T1 counterparts, and four-rule first-order rule IT2 partitions are much larger in size than their T1 counterparts, which can again be summarized by stating: *It seems that MF uncertainty tends to fire more rules more often* (at least in this example).

3) First-Order Rule Partitions for $X_1 \times X_2 \times \cdots \times X_p$:

Definition 11: In a T1 (IT2) fuzzy system, a *T1 (IT2) first-order rule partition* of $X_1 \times X_2 \times \cdots \times X_p$ is a collection of non-overlapping hyperrectangles (or squares) in $X_1 \times X_2 \times \cdots \times X_p$, in each of which the *same* number of *same* rules is fired whose firing levels (intervals) contribute to the output of a T1 (IT2) fuzzy system.

Notations for T1 (IT2) first-order rule partitions of $X_1 \times X_2 \times \cdots \times X_p$ are in the bottom half of Table I. Note that

$$N_*^1(X_1, X_2, \ldots, X_p) = \prod_{i=1}^{p} N_*^1(X_i) \tag{6}$$

$$N_R(k_{x_1}, k_{x_2}, \ldots, k_{x_p}) = \prod_{i=1}^{p} N_R(k_{x_i}). \tag{7}$$

Returning to Example 2, observe from (6) that $N_{\text{T1}}^1(X_1, X_2) = N_{\text{IT2}}^1(X_1, X_2) = 5 \cdot 5 = 25$, which is in agreement with the direct counts that were obtained from Fig. 4(a) and (b).

[8]There are many ways to "blur" the Fig. 3(a) MFs to arrive at their IT2 FOUs (see the first three parts of Example 3 in Section I of the Supplementary Material). We have assumed that there is uncertainty about *a*, *b*, *c*, and *d* to arrive at the FOUs in Fig. 3(b), and leave it to the reader to examine other choices.

[9]If a rule has p antecedents, then X_1 and X_2 each denote the universe of discourse for any two of them.

TABLE IV
NOTATION USED FOR SECOND-ORDER RULE PARTITIONS. SUBSCRIPT $*$ REFERS TO EITHER T1 OR IT2

Second-Order Rule Partitions	
Symbol	Definition $(k_{x_i} = 1, \ldots, N_*^1(x_i))$
$P_*^2(k_{x_i}, m_{k_{x_i}} \| x_i)$	T1 or IT2 second-order rule partition of X_i, often abbreviated to $(k_{x_i}, m_{k_{x_i}})$
$m_{k_{x_i}}$	Counter/index of T1 or IT2 second-order rule partition of X_i; $m_{k_{x_i}} = 1, \ldots, N_*^2(k_{x_i} \| x_i)$
$N_*^2(k_{x_i} \| x_i)$	Total number of T1 or IT2 second-order rule partitions within T1 or IT2 first-order rule partition of X_i, $P_*^1(k_{x_i} \| x_i)$
$N_*^2(X_i)$	Total number of T1 or IT2 second-order rule partitions of X_i
$P_*^2((k_{x_1}, k_{x_2}), m_{(k_{x_1}, k_{x_2})})$	T1 or IT2 second-order rule partition of $X_1 \times X_2$
$m_{(k_{x_1}, k_{x_2})}$	Counter/index of T1 or IT2 second-order rule partition of $X_1 \times X_2$; $m_{(k_{x_1}, k_{x_2})} = 1, \ldots, N_*^2(k_{x_1}, k_{x_2})$
$N_*^2(k_{x_1}, k_{x_2})$	Total number of T1 or IT2 second-order rule partitions within the (k_{x_1}, k_{x_2})th T1 or IT2 first-order rule partition of $X_1 \times X_2$
$Z(X_i)$	Number of times that $N_*^2(k_{x_i} \| x_i) = 0$
$N_*^{2\prime}(X_i)$	$N_*^2(X_i) + Z(X_i)$
$N_*^2(X_1, X_2, \ldots, X_p)$	Total number of T1 or IT2 second-order rule partitions of $X_1 \times X_2 \times \cdots \times X_p$ [use (13)]

Even for rules that have $p \geq 4$ antecedents, for which it is not possible to visualize the T1 (IT2) first-order rule partitions, it is still possible to compute their total number by (6) as well as the fixed number of rules that are fired in each hyperrectangle by (7).

It should be evident by now that it is first-order uncertainty partitions that are responsible for T1 first-order rule partitions and second-order uncertainty partitions that are responsible for IT2 first-order rule partitions, and that *it is such T1 (IT2) first-order rule partitions that make a T1 (IT2) fuzzy system uniquely different from a crisp system*. But, there is much more!

B. Second-Order Rule Partitions in a Fuzzy System

When a MF (FOU) changes its mathematical formula within a first-order rule partition, second-order rule partitions are obtained.

1) Second-Order Rule Partitions for Each X_i:

Definition 12: In a T1 (IT2) fuzzy system, a *T1 (IT2) second-order rule partition of* X_i occurs when the MF (FOU) of a T1 (IT2) fuzzy set that is associated with x_i changes its mathematical formula within a T1 (IT2) first-order rule partition of X_i.

Notations for T1 (IT2) second-order rule partitions are given in Table IV. Unfortunately, they are more complicated than the notations for first-order rule partitions because there are now two indexes: k_{x_i} indexes the first-order partitions and $m_{k_{x_i}}$ indexes the second-order partitions that occur within k_{x_i}.

A formal four-step procedure for establishing $P_*^2(k_{x_i}, m_{k_{x_i}} | x_i)$, $N_*^2(k_{x_i} | x_i)$, and $N_*^2(X_i)$ begins with a plot (sketch) of the first-order rule partitions and proceeds as in Table V. $Z(x_i)$ and $N_*^{2\prime}(X_i)$ are explained in Section B.3.

TABLE V
FOUR-STEP PROCEDURE FOR ESTABLISHING T1 (IT2) SECOND-ORDER RULE PARTITION QUANTITIES FOR A SINGLE VARIABLE x_i, IN A T1 (IT2) FUZZY SYSTEM, ON A PLOT (SKETCH) OF ITS RESPECTIVE FIRST-ORDER RULE PARTITIONS

Step	Description
1	Scan the axis of x_i with an imaginary dotted vertical line from left to right. Wherever a MF (LMF or UMF) changes its formula, draw a dotted vertical line. If the change in formula occurs at a boundary of a T1 (IT2) first-order rule partition, then do not draw such a vertical dotted line.
2	The interval of real numbers between adjacent dotted vertical lines or between a dotted line and a dashed (or dashed and dotted) line is its T1 (IT2) second-order rule partition $[P_*^2(k_{x_i}, m_{k_{x_i}} \| x_i)]$.
3	Each T1 (IT2) first-order rule partition has from zero to a finite number of T1 (IT2) second-order rule partitions $[N_*^2(k_{x_i} \| x_i)]$.
4	Count the total number of $N_*^2(k_{x_i} \| x_i)$, the total being $N_*^2(X_i)$.

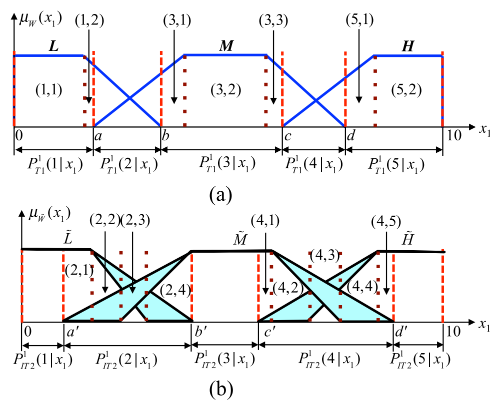


Fig. 5. Example 3 figures: (a) T1 FSs and (b) IT2 FSs.

Example 3: This is a continuation of Example 1. The results for Table V's Steps 1–3 are shown in Fig. 5(a) and (b). For the T1 fuzzy system, on Fig. 5(a): $P_{\mathrm{T1}}^1(1|x_1)$ has $N_{\mathrm{T1}}^2(1|x_1) = 2$ T1 second-order rule partitions that are labeled $(1,1)$ and $(1,2)$; $P_{\mathrm{T1}}^1(2|x_1)$ and $P_{\mathrm{T1}}^1(4|x_1)$ have zero T1 second-order rule partitions; $P_{\mathrm{T1}}^1(3|x_1)$ has $N_{\mathrm{T1}}^2(3|x_1) = 3$ T1 second-order rule partitions that are labeled $(3,1)$, $(3,2)$, and $(3,3)$; and, $P_{\mathrm{T1}}^1(5|x_1)$ has $N_{\mathrm{T1}}^2(5|x_1) = 2$ T1 second-order rule partitions that are labeled $(5,1)$ and $(5,2)$; hence, $N_{\mathrm{T1}}^2(X_1) = \sum_{k_{x_1}=1}^{5} N_{\mathrm{T1}}^2(k_{x_1}|x_1) = 7$.

For the IT2 fuzzy system, in Fig. 5(b): $P_{\mathrm{IT2}}^1(1|x_1)$, $P_{\mathrm{IT2}}^1(3|x_1)$, and $P_{\mathrm{IT2}}^1(5|x_1)$ have zero second-order rule partitions; $P_{\mathrm{IT2}}^1(2|x_1)$ has $N_{\mathrm{IT2}}^2(2|x_1) = 4$ IT2 second-order rule partitions that are labeled $(2,1), \ldots, (2,4)$; and, $P_{\mathrm{IT2}}^1(4|x_1)$ has $N_{\mathrm{IT2}}^2(4|x_1) = 5$ IT2 second-order rule partitions that are labeled $(4,1), \ldots, (4,5)$; hence, $N_{\mathrm{IT2}}^2(X_1) = \sum_{k_{x_1}=1}^{5} N_{\mathrm{IT2}}^2(k_{x_1}|x_1) = 9$.

Generally speaking, an IT2 fuzzy system has more second-order rule partitions than its comparable T1 fuzzy system, because it uses two sets of T1 FSs, LMFs and UMFs.

2) Second-Order Rule Partitions for $X_1 \times X_2$:

Definition 13: In a T1 (IT2) fuzzy system, a *T1 (IT2) second-order rule partition of* $X_1 \times X_2$ occurs when the MF (FOU) of a T1 (IT2) fuzzy set that is associated with *either* x_1 *or* x_2

TABLE VI
FOUR-STEP PROCEDURE FOR ESTABLISHING T1 (IT2) SECOND-ORDER RULE PARTITION QUANTITIES FOR $X_1 \times X_2$ IN A T1 (IT2) FUZZY SYSTEM, ON A PLOT (SKETCH) OF THE FIRST-ORDER RULE PARTITIONS

Step	Description
1	Locate the T1 (IT2) second-order rule partitions of $x_1 (x_2)$ on the horizontal (vertical) axis.
2	Extend all dotted T1 second-order rule partitions so that they cover $X_1 \times X_2$. The results from doing this will be a collection of rectangles (or squares).
3	Each T1 (IT2) first-order rule partition on $X_1 \times X_2$ has from zero to a finite number of T1 (IT2) second-order rule partitions. Establish $N_*^2(k_{x_1}, k_{x_2})$ by counting.
4	Count the total number of $N_*^2(k_{x_1}, k_{x_2})$, the total being $N_*^2(X_1, X_2)$.

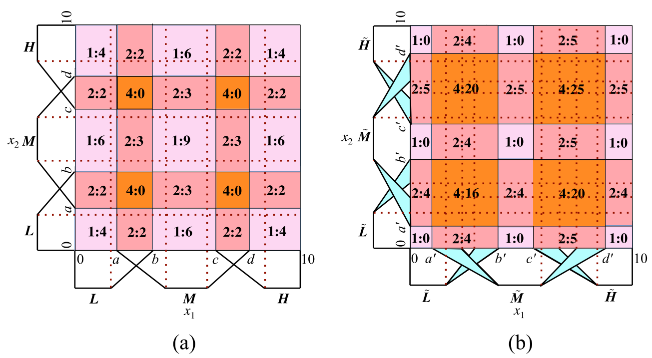


Fig. 6. Example 4 figures for Table VI's Steps 1–3: (a) T1 FSs and (b) IT2 FSs.

changes its mathematical formula within a T1 (IT2) first-order rule partition of $X_1 \times X_2$.

Notations for T1 (IT2) second-order rule partitions of $X_1 \times X_2$ are in the second portion of Table IV. Note that

$$N_*^2(X_1, X_2) = \sum_{k_{x_2}=1}^{N_*^1(X_2)} \sum_{k_{x_1}=1}^{N_*^1(X_1)} N_*^2(k_{x_1}, k_{x_2}). \tag{8}$$

In order to use (8) a formula is needed for $N_*^2(k_{x_1}, k_{x_2})$. Instead of providing such a localized formula, we take a different approach in Section B.3 to computing $N_*^2(X_1, X_2)$.

A formal four-step procedure for establishing $P_*^2((k_{x_1}, k_{x_2}), m_{(k_{x_1}, k_{x_2})})$, $N_*^2(k_{x_1}, k_{x_2})$, and $N_*^2(X_1, X_2)$ begins with a plot (sketch) of the first-order rule partitions and proceeds as in Table VI.

Example 4: This is a continuation of Examples 2 and 3. The results of Steps 1–3 of Table VI are shown on Fig. 6(a) and (b). In order to keep these figures as uncluttered as possible, we have left off the detailed notations like the ones that are in Fig. 5(a) and (b). In each first-order rule partition there are two numbers that are separated by a colon; the first is $N_R(k_{x_1}, k_{x_2})$ and the second is $N_*(k_{x_1}, k_{x_2})$, e.g., 2:3 indicates that two rules are fired in a first-order rule partition and there are three second-order rule partitions in that first-order rule partition.

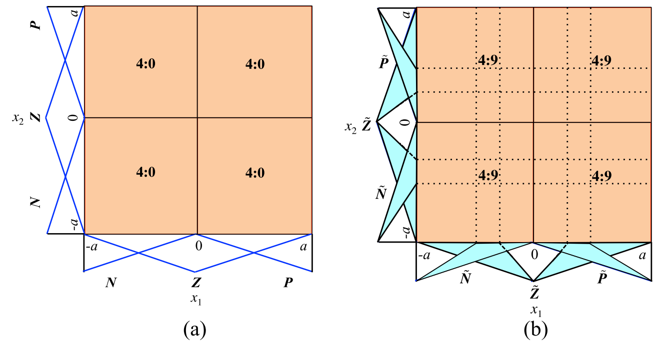


Fig. 7. Example 5 figures: (a) T1 FSs and (b) IT2 FSs.

By adding all of the numbers that appear to the right of the colons in Fig. 6(a), one finds that

$$N_{\mathrm{T1}}^2(X_1, X_2) = \sum_{k_{x_2}=1}^{5} \sum_{k_{x_1}=1}^{5} N_{\mathrm{T1}}^2(k_{x_1}, k_{x_2}) = 77. \tag{9}$$

So, the T1 fuzzy system with only 9 rules and 25 T1 first-order rule partitions has 77 T1 second-order rule partitions, which demonstrates a very large amount of sculpting on $X_1 \times X_2$, something that cannot be achieved with a crisp 9-rule rule-based system.

By adding all of the numbers that appear to the right of the colons in Fig. 6(b), one finds that

$$N_{\mathrm{IT2}}^2(X_1, X_2) = \sum_{k_{x_2}=1}^{5} \sum_{k_{x_1}=1}^{5} N_{\mathrm{IT2}}^2(k_{x_1}, k_{x_2}) = 135. \tag{10}$$

So, the IT2 fuzzy system, which has the same number of T1 first-order rule partitions as the T1 fuzzy system, has close to twice as many IT2 second-order rule partitions. Consequently, it is able to provide a much greater and finer sculpting of $X_1 \times X_2$ than is the T1 fuzzy system.

Example 5: This is another nine-rule fuzzy system in which $x_1, x_2 \in [-a, a]$. Fig. 7(a) depicts the T1 first- and second-order rule partitions. Observe that there are four T1 first-order rule partitions each of which has four fired rules. Observe, also, that there are no T1 second-order rule partitions.

Fig. 7(b) depicts the IT2 first- and second-order rule partitions. Observe that there are also four IT2 first-order rule partitions each of which also has four fired rules, but each IT2 first-order rule partition contains nine IT2 second-order rule partitions. Consequently, although the T1 and IT2 fuzzy systems have exactly the same number of first-order rule partitions (which in this example are less than the total number of rules), there is no further sculpting of the T1 fuzzy system, whereas there is much further sculpting of the IT2 fuzzy system.

3) Second-Order Rule Partitions for $X_1 \times X_2 \times \cdots \times X_p$: Definition 14: In a T1 (IT2) fuzzy system, a *T1 second-order rule partition of* $X_1 \times X_2 \times \cdots \times X_p$ occurs when the MF (FOU) of the T1 (IT2) fuzzy set associated with *either* x_1, *or* x_2, *or* $\ldots$, *or* x_p changes its mathematical formula within a T1 (IT2) first-order rule partition of $X_1 \times X_2 \times \cdots \times X_p$.

Of course, it is impractical (impossible) to extend the Table VI procedure from two to more than three variables (even for three variables the construction would be very tedious). What is arguably most important is to determine the total number of T1 (IT2) second-order rule partitions without having to directly count the number of such partitions in each T1 (IT2) first-order rule partition.

Interesting Observations: In Fig. 6(a) and (b) observe that it is only in regions of $X_1 \times X_2$ where *both* x_1 and x_2 individually have no second-order rule partitions that such regions also have no second-order rule partitions. There are four such regions in Fig 6(a), two of which are $[a, b] \times [a, b]$ and $[a, b] \times [c, d]$, and there are nine such regions in Fig. 6(b), two of which are $[0, a'] \times [0, a']$ and $[b', c'] \times [b', c']$. In such regions, multiplying 0 by 0 gives the correct number of second-order rule partitions, which is also 0. On the other hand, regions of $X_1 \times X_2$ where either (but not both) x_1 or x_2 individually have no second-order rule partitions jointly have a *nonzero* number of second-order rule partitions. There are six such regions in Fig. 6(a), three of which are $[a, b] \times [0, a]$, $[a, b] \times [b, c]$, and $[c, d] \times [d, 10]$, and six such regions in Fig. 6(b), three of which are $[0, a'] \times [a', b']$, $[0, a'] \times [c', d']$, and $[d', 10] \times [c', d']$. In such regions, multiplying 0 by any nonzero number always gives 0, which is not the correct number of the region's second-order rule partitions. If, instead the 0 is replaced by 1 then multiplying 1 by a nonzero number gives the correct number of the region's second-order rule partitions.

These observations lead to the following novel way to compute $N_*^2(X_1, X_2)$: Let the number of times that $(i = 1, 2)$ $N_*^2(k_{x_i}|x_i) = 0$ be called $Z(X_i)$, and let

$$N_*^{2\prime}(X_i) \equiv N_*^2(X_i) + Z(X_i). \tag{11}$$

Then

$$N_*^2(X_1, X_2) = N_*^{2\prime}(X_1) N_*^{2\prime}(X_2) - Z(X_1) Z(X_2). \tag{12}$$

Example 6: This is a continuation of Examples 3 and 4. For the T1 fuzzy system, applying the Example 3 results to both x_1 and x_2, $N_{\mathrm{T1}}^2(X_1) = N_{\mathrm{T1}}^2(X_2) = 7$. Examining Fig. 5(a), it is clear that $Z(x_1) = Z(x_2) = 2$. Consequently, using (11) and then (12), one finds $N_{\mathrm{T1}}^{2\prime}(X_1) = N_{\mathrm{T1}}^{2\prime}(X_2) = 9$ and $N_{\mathrm{T1}}^2(X_1, X_2) = 9 \cdot 9 - 2 \cdot 2 = 77$, which agrees with $N_{\mathrm{T1}}^2(X_1, X_2)$ that was obtained in Example 4's (9).

For the IT2 fuzzy system, applying the Example 3 results to both x_1 and x_2, $N_{\mathrm{IT2}}^2(X_1) = N_{\mathrm{IT2}}^2(X_2) = 9$. Examining Fig. 5(b), it is clear that $Z(x_1) = Z(x_2) = 3$. Consequently, using (11) and then (12), one finds $N_{\mathrm{IT2}}^{2\prime}(X_1) = N_{\mathrm{IT2}}^{2\prime}(X_2) = 12$ and $N_{\mathrm{IT2}}^2(X_1, X_2) = 12 \cdot 12 - 3 \cdot 3 = 135$, which agrees with $N_{\mathrm{IT2}}^2(X_1, X_2)$ that was obtained in Example 4's (10).

Due to space limitations, using (11) and (12) for Example 5 are left to the reader.

The extension of (12) to p variables is

$$N_*^2(X_1, X_2, \ldots, X_p) = \prod_{i=1}^{p} N_*^{2\prime}(X_i) - \prod_{i=1}^{p} Z(X_i). \tag{13}$$

The sculpting of $X_1 \times X_2 \times \cdots \times X_p$ by means of second-order rule partitions grows (roughly) exponentially with *p* both for T1 and IT2 fuzzy systems; however, if each variable in the IT2 fuzzy system has more IT2 second-order rule partitions than it does for a T1 fuzzy system, then the sculpting of $X_1 \times X_2 \times \cdots \times X_p$ will be vastly greater for the IT2 fuzzy system.

4) Observations: Some additional examples of T1 and IT2 rule partitions are given in the Supplementary Material to this paper. From these examples, as well as the earlier examples, for which MFs are piece-wise linear, it can be observed that:

1) In many examples, the number of first-order rule partitions for x_i equals 2 + the number of[10] MFs (or UMFs) that do not span the entire universe of discourse (as Z does in Fig. 7). Hence, increasing the number of first-order rule partitions can be achieved by granulating x_i more finely into more fuzzy sets.
2) The point at which a MF changes its mathematical formula is called a *kink*. The number of second-order rule partitions in a first-order rule partition of x_i, when there is at least one MF kink in the latter, equals 1 + the number of MF kinks in the first-order rule partition; when there are zero MF kinks in the first-order rule partition, there are zero second-order rule partitions; hence, greater sculpting is achieved by using MFs (UMFs and LMFs) that have more kinks.
3) Because an IT2 fuzzy set is described by two T1 fuzzy sets (LMF and UMF), it always has the potential to have more second-order rule partitions than a T1 fuzzy set; hence, an IT2 fuzzy system almost always has the potential to out-sculpt a T1 fuzzy system when both use the same number of MFs (FOUs) for each variable. Why this is stated as "almost always" rather than "always" is explained in Section IV.

IV. Novelty Partitions

So far, our attention has been directed exclusively at the partitioning of $X_1 \times X_2 \times \cdots \times X_p$ due to the interactions of inputs to a fuzzy system with their respective antecedents. *Each T1 or IT2 second-order rule partition contains a NL system where the exact nature of the nonlinearity depends on rule consequents and output processing*. In both T1 and IT2 fuzzy systems that use direct defuzzification there is no further partitioning of $X_1 \times X_2 \times \cdots \times X_p$; but, in IT2 fuzzy systems that use TR there is another layer of partitioning of $X_1 \times X_2 \times \cdots \times X_p$ into *IT2 novelty partitions*.

Definition 15: IT2 novelty[11] *partitions* of $X_1 \times X_2 \times \cdots \times X_p$ occur only when TR is used, and result from different end-points of a firing interval being used to compute the end-points of the type-reduced set. They occur within an IT2 first-order rule partition, regardless of whether or not there are any IT2 second-order rule partitions.

It is very difficult to determine and display the geometry of IT2 novelty partitions because there are no closed-form formulas for the end-points of a type-reduced set. This does not diminish their importance because they provide us with

[10] T1 fuzzy partitions are excluded herein because there are no IT2 counterparts to them.

[11] *Novelty*, as defined here, was introduced first in [50] and [51].

TABLE VII
RULE BASE OF THE IT2 FPID CONTROLLER

$E/\Delta E$	$\tilde{N}_{\Delta E}$	$\tilde{Z}_{\Delta E}$	$\tilde{P}_{\Delta E}$
$\tilde{N}_E$	$\tilde{R}_Z^1 : U = -1$	$\tilde{R}_Z^2 : U = -0.5$	$\tilde{R}_Z^3 : U = 0$
$\tilde{Z}_E$	$\tilde{R}_Z^4 : U = -0.5$	$\tilde{R}_Z^5 : U = 0$	$\tilde{R}_Z^6 : U = 0.5$
$\tilde{P}_E$	$\tilde{R}_Z^7 : U = 0$	$\tilde{R}_Z^8 : U = 0.5$	$\tilde{R}_Z^9 : U = 1$

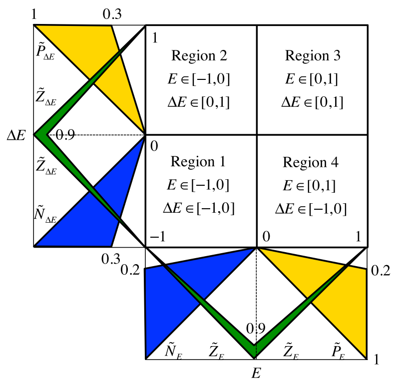
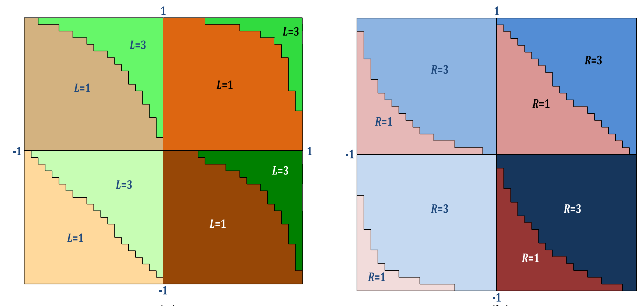


Fig. 8. IT2 first-order rule partitions for the IT2 FPID controller.



(a) (b)

Fig. 9. IT2 novelty partitions for (a) y_l^{COS} and (b) y_r^{COS} in the IT2 FPID controller. Switch points L and R correspond to L and R in (2) and (3), respectively.

additional insight into the further partitioning (sculpting) of $X_1 \times X_2 \times \cdots \times X_p$, something that can only occur for an IT2 fuzzy system that uses TR, and *can never occur in a T1 fuzzy system*. IT2 novelty partitions may help to explain why system performance is often arguably better for an IT2 fuzzy system that uses TR than it is for one that does not use TR.

Some examples of IT2 novelty partitions that use the minimum t-norm[12] are in [38, Ch. 4 and 5)], [15], [41], [58]. In the rest of this section we shall display the geometry of IT2 novelty partitions (for the product t-norm and COS TR) by using portions of a case study that is in[13] [37, Ch. 10]. Our focus is on an IT2 fuzzy PID (FPID) controller, U, that has two normalized inputs, E and $\dot{E} \equiv \Delta E$. It uses the symmetrical 3×3 rule base in Table VII. The rule structure of the IT2 FPID controller is $(l = 1, \ldots, 9)$

$$\tilde{R}_Z^l : \text{IF } E \text{ is } \tilde{F}_1^l \text{ and } \Delta E \text{ is } \tilde{F}_2^l \text{ THEN } U \text{ is } G^l. \tag{14}$$

In (14), both E and ΔE are described by the three overlapping FOUs that are depicted in Fig. 8, and G^l are the crisp singletons that are tabulated in Table VII.

It should be clear, from Fig. 8, that E and ΔE each have two IT2 first-order rule partitions in which two rules are fired, and that there are no IT2 second-order rule partitions. In fact, a T1 FPID controller that uses the UMFs of the three T2 FSs will also have two T1 first-order rule partitions in which two rules are fired, and no T1 second-order rule partitions. These facts make this a very interesting example because the playing field has been leveled for both the T1 and IT2 FPID controllers in terms of first- and second-order rule partitions.

Fig. 8 depicts the IT2 first-order rule partitions for $E \times \Delta E$ as four regions in each of which the IT2 FPID controller uses four rules. Because the LMFs and UMFs are linear, it is easy to write formulas for them, after which it is then easy to write formulas for the lower and upper ends of the firing intervals in each region. Table SM-1 in the Supplementary Material tabulates all of this information for each of the four regions. This is such a relatively simple example that, instead of using any of the iterative algorithms to compute the switch points of the COS type-reduced set, it is more instructive to use brute force by considering the five possible iterations that are summarized in Table SM-2 in the Supplementary Material.

Fig. 9 depicts the IT2 novelty partitions for y_l^{COS} and y_r^{COS}. Observe that each of these four regions has two IT2 novelty partitions and, therefore, $E \times \Delta E$ has eight IT2 novelty partitions for both y_l^{COS} and y_r^{COS}. When y_l^{COS} and y_r^{COS} are combined during defuzzification this will involve an overlay of the two sets of IT2 novelty partitions, thereby partitioning (sculpting) $E \times \Delta E$ into 16 IT2 novelty partitions. *None of this can occur in a T1 FPID controller or in an IT2 FPID controller that does not use* TR. These IT2 novelty partitions may help to explain why of four IT2 FPID controllers that were designed and reported in [37, Ch. 10], the[14] one that gave the best control system performance metrics[15] used COS TR.

Control surfaces for T1 and four IT2 FPID controllers are given in [37, Fig. 10.24], and Fig. SM-1 in the Supplementary Material. They clearly reveal the more undulating surface for

[12]Determining novelty partitions, when the minimum t-norm is used, is more difficult to do then when the product t-norm is used because the minimum of two quantities is a conditional result, whereas the product is not.

[13]The case study was created by Prof. T. Kumbasar, most generously.

[14]The four are (e.g., [37]): (1) COS TR + defuzzification; (2) Nie-Tan direct defuzzification; (3) Biglarbegian, Melek and Mendel direct defuzzification; and, (4) WM uncertainty bounds direct defuzzification. The latter three involve no TR and consequently have no IT2 novelty partitions. From a rule-partition perspective, all four have 4 IT2 first-order rule partitions and no IT2 second-order rule partitions; so they are all variable-structure NL systems, each with 4 NL subsystems. But, because of COS TR, the first design has 16 IT2 novelty partitions, and so it is a variable-structure NL system with 16 NL subsystems.

[15]The performance metrics are: % overshoot, settling time and integral absolute error. "Best control system performance metrics" means smaller numerical values for these metrics.

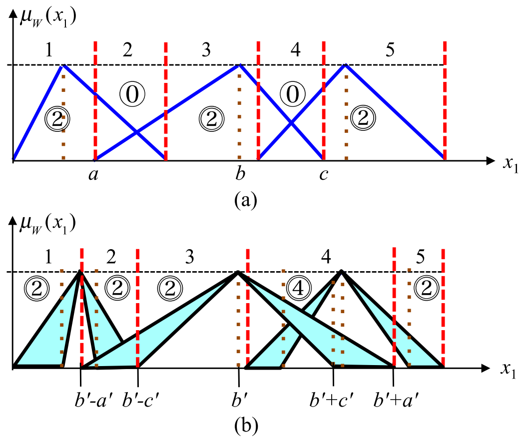


Fig. 10. Example 7 figures: (a) T1 FSs and (b) IT2 FSs. Numbers above top dashed line denote the index of the T1 or IT2 first-order rule partition; circled numbers denote the number of T1 or IT2 second-order rule partitions contained within a first-order rule partition.

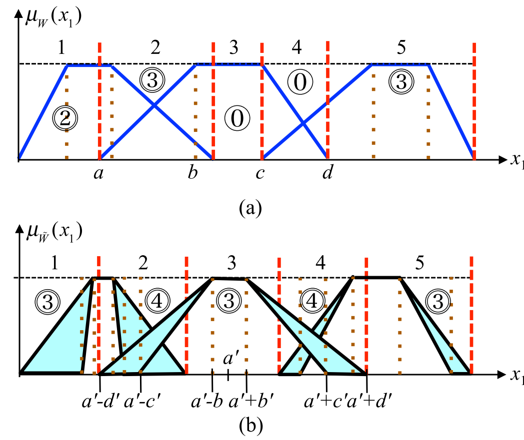


Fig. 11. Example 8 figures: (a) T1 FSs and (b) IT2 FSs. The second sentence in the caption of Fig. 9 applies here as well.

the COS + defuzzification IT2 FPID controller, which for this controller is only due to the sculpting of $E \times \Delta E$ by IT2 novelty partitions.

V. Gaussian MFs and FOUs

Gaussian MFs are sometimes used in T1 fuzzy systems and frequently used in IT2 fuzzy systems. Three reasons for the latter are:

1) because each such MF (Gaussian LMF and UMF) spans the entire universe of discourse, the resulting output surfaces are always continuous (this is proved in [53]);
2) it is straightforward to compute derivatives of such MFs (or LMFs and UMFs) with respect to their parameters when such derivatives are used in gradient-based optimization algorithms;
3) such MFs and FOUs are emphasized in [35], which is very highly referenced by T2 fuzzy system researchers.

Unfortunately, because Gaussian MFs (FOUs) span the entire universe of discourse, fuzzy systems that use them have reduced sculpting capabilities, i.e., such fuzzy systems (T1 or IT2) have one first-order rule partition (i.e., all rules fire over all of $X_1 \times X_2 \times \cdots \times X_p$, although many of the firing quantities will be very small) and none to some second-order rule partitions.[16] However, an IT2 fuzzy system that uses COS TR + defuzzification will still have IT2 novelty partitions.

These observations may help to explain why when comparison studies are made between T1 and IT2 fuzzy systems that both use Gaussian T1 MFs, LMFs, and UMFs, one does not see much performance improvement for the IT2 fuzzy system, and suggests that the choice made for the shapes of T1 and IT2 MFs or FOUs is more important than has been believed in the past. Rule and IT2 novelty partitions suggest that MFs or FOUs that have greater sculpting capabilities will lead to fuzzy systems that can outperform comparable fuzzy systems that have lesser sculpting capabilities.

[16] T1 Gaussians and IT2 Gaussians with uncertain standard deviations have none; but, IT2 Gaussians with uncertain means have some. Additionally, truncated Gaussians will have more first- and second-order rule partitions; however, the greater the accuracy the less the truncation, and consequently there will be fewer first- and second-order rule partitions.

VI. Equality of Parameters and Sculpting

Here we choose MFs and FOUs that have exactly the same number of parameters and demonstrate the advantage that the IT2 fuzzy system still has in terms of its greater sculpting capabilities. We do this by means of two examples.

Example 7: In Fig. 10 the T1 MFs are asymmetric triangles, each described by three parameters, e.g., a, b, and c, whereas the IT2 FOUs have symmetrical LMFs and UMFs collectively described by three parameters, e.g., a', b', and c'. Although x_1 has 5 first-order rule partitions for both T1 and IT2 fuzzy sets, it only has 6 T1 second-order rule partitions, whereas it has 12 IT2 second-order rule partitions. Hence, even though the T1 and IT2 fuzzy sets have the same number of parameters, the IT2 fuzzy sets have twice as much sculpting capability. And, of course, when COS TR + defuzzification is used for the IT2 fuzzy system, that system will also have IT2 novelty partitions that will further sculpt its state space.

Example 8: In Fig. 11 the T1 MFs are asymmetric trapezoids, each described by four parameters, e.g., a, b, c, and d, whereas the IT2 FOUs have symmetrical trapezoid LMFs and UMFs, so each FOU is also described by four parameters, e.g., a', b', c', and d'. Although x_1 again has 5 first-order rule partitions for both T1 and IT2 fuzzy sets, it only has 8 T1 second-order rule partitions, whereas it has 17 IT2 second-order rule partitions. Hence, even though the T1 and IT2 fuzzy sets have the same number of parameters, the IT2 fuzzy sets again have significantly more sculpting capability. And, of course again, when COS TR + defuzzification is used for the IT2 fuzzy system, it will also have IT2 novelty partitions that further sculpt its state space.

Although examples are not a rigorous proof, these two, which are representative of a multitude of other examples, clearly show that even when T1 and IT2 fuzzy systems are described by exactly the same number of parameters, the IT2 fuzzy system has much greater sculpting capabilities than the T1 fuzzy system, which can provide the IT2 fuzzy system with an ability to outperform the T1 fuzzy system.

VII. Conclusion and Future Research

This paper has provided some new and novel application-independent perspectives on why improved performance usually occurs as one goes from crisp, to T1, and then to IT2 fuzzy systems, using three kinds of partitions.

Uncertainty partitions let T1 fuzzy sets be distinguished from crisps sets, IT2 fuzzy sets be distinguished from T1 fuzzy sets, and even GT2 fuzzy sets be distinguished from IT2 fuzzy sets.

Rule partitions are a direct result of uncertainty partitions and are associated with firing levels for T1 fuzzy systems, and firing intervals for IT2 fuzzy systems. *First-order rule partitions* partition (sculpt) $X_1 \times X_2 \times \cdots \times X_p$ into hyperrectangles within which are contained the *same* number of fired rules. It is not uncommon for T1 and IT2 fuzzy systems to have the same number of first-order rule partitions, although MF uncertainties tend to fire more rules more often in the IT2 fuzzy systems. *Second-order rule partitions* can occur when MFs change their mathematical formulae within a first-order rule partition. They sculpt $X_1 \times X_2 \times \cdots \times X_p$, more finely, and it is very common for an IT2 fuzzy system to have many (vastly) more second-order rule partitions than a T1 fuzzy system.

Additionally, an IT2 fuzzy system that uses COS TR $+$ defuzzification further partitions $X_1 \times X_2 \times \cdots \times X_p$ into more hyperregions, called IT2 *novelty partitions*, and this provides such IT2 fuzzy systems with an even greater sculpting of $X_1 \times X_2 \times \cdots \times X_p$. IT2 novelty partitions overlay IT2 second-order rule partitions, or, if there are no IT2 second-order rule partitions, they overlay IT2 first-order rule partitions. And, IT2 novelty partitions can never occur in T1 fuzzy systems because TR does not occur for them.

Within each sculpted partition of $X_1 \times X_2 \times \cdots \times X_p$ is an NL fuzzy subsystem, and even though a T1 fuzzy system may be described by a large number of such NL subsystems, an IT2 fuzzy system that uses COS TR $+$ defuzzification will always be described by more of them, and its NL subsystems are always more NL than those of the T1 fuzzy system. It is the author's conjecture that: *it is the greater sculpting of $X_1 \times X_2 \times \cdots \times X_p$ that usually lets an IT2 fuzzy system outperform a T1 fuzzy system, and usually lets a T1 fuzzy system outperform a crisp system.*

Some open research questions and extensions to this paper are:

1) prove the just-stated conjecture using the framework of rule and IT2 novelty partitions;
2) re-examine the value of choosing Gaussian MFs or FOUs using the framework of rule and IT2 novelty partitions;
3) examine if mathematical formulas can be obtained for IT2 novelty partitions, or will we always be limited to obtaining them by computer simulations?;
4) extend the results to nonsingleton fuzzifiers and to other kinds of T1 and IT2 fuzzy systems, such as intuitionistic and hesitant fuzzy systems, to explain *a priori* if they lead to even more partitions (sculpting), thereby suggesting that they will outperform fuzzy systems that only use traditional T1 and IT2 fuzzy sets;
5) extend the results to GT2 fuzzy systems, using the horizontal-slice (also known as the $\alpha-$ plane [30] or zSlice [48]) decomposition of a GT2 fuzzy set, to understand *a priori* whether or not the third dimension of such fuzzy sets leads to even more partitions of $X_1 \times X_2 \times \cdots \times X_p$;
6) extend the results to other t-norms;
7) extend the results to T1 and IT2 rule-based classifiers (e.g., [27]);
8) extend the results to T1 and IT2 clustering (e.g., [3], [42]);
9) develop an *inverse rule partition theory* in which one begins by specifying the geometrical natures of some critical first- and second-order rule hyperpartitions in $X_1 \times X_2 \times \cdots \times X_p$ and then works backward to establish the MFs or FOUs that lead to such partitions.

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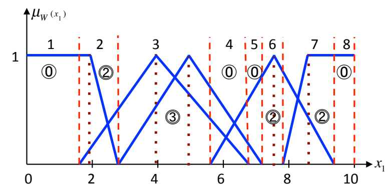
Supplementary Materials

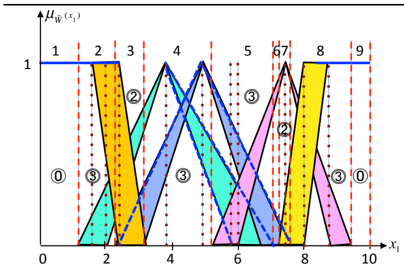
I. More Rule-Partition Examples

In the figures for all three examples of this section, numbers above the MFs or FOUs denote the index of the T1 or IT2 first-order rule partitions, and circled numbers denote the number of T1 or IT2 second-order rule partitions that are contained within a first-order rule partition. Tables that are to the right of the figures give numerical values for k_{x_1}, $N_R(k_{x_1})$ and $N_*^2(k_{x_1} \mid x_1)$.

Example 1: This is an example where three rules fire. For the T1 fuzzy system this occurs when $k_{x_1} = 4$, and for the IT2 fuzzy system this occurs when $k_{x_1} = 3$, 5 and 7. This example demonstrates that the number of first-order rule partitions does not have to be the same for T1 and IT2 fuzzy systems, and that MF uncertainty can fire more numbers of rules more often. The T1 fuzzy system has 9 T1 second-order rule partitions, whereas the IT2 fuzzy system has 16 IT2 second-order rule partitions.

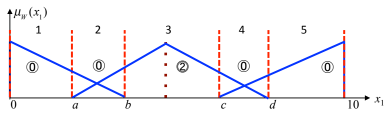


k_{x_1}	1	2	3	4	5	6	7	$8 = N^1_{T1}(X_1)$	
$N_R(k_{x_1})$	1	2	2	3	2	1	2	1	
$N^2_{T1}(k_{x_1} \mid x_1)$	0	2	3	0	0	2	2	0	$N^2_{T1}(X_1) = 9$

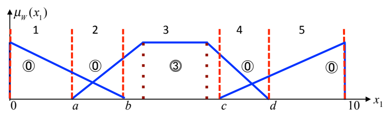


k_{x_1}	1	2	3	4	5	6	7	8	$9 = N^1_{IT2}(X_1)$	
$N_R(k_{x_1})$	1	2	3	2	3	2	3	2	1	
$N^2_{IT2}(k_{x_1} \mid x_1)$	0	3	2	3	3	0	2	3	0	$N^2_{IT2}(X_1) = 16$

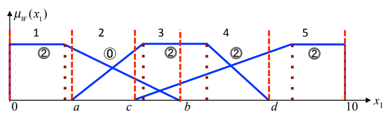
Example 2: More examples of T1 first- and second-order rule partitions. These examples illustrate what happens to the number of first- and second-order rule partitions as some or all of the MFs change shape.



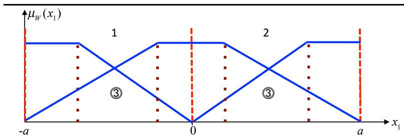
k_{x_1}	1	2	3	4	$5 = N_{T1}^1(X_1)$	
$N_R(k_{x_1})$	1	2	1	2	1	
$N_{T1}^2(k_{x_1} \mid x_1)$	0	0	2	0	0	$N_{T1}^2(X_1) = 2$



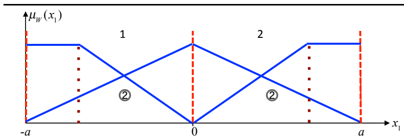
k_{x_1}	1	2	3	4	$5 = N_{T1}^1(X_1)$	
$N_R(k_{x_1})$	1	2	1	2	1	
$N_{T1}^2(k_{x_1} \mid x_1)$	0	0	3	0	0	$N_{T1}^2(X_1) = 3$



k_{x_1}	1	2	3	4	$5 = N_{T1}^1(X_1)$	
$N_R(k_{x_1})$	1	2	3	2	1	
$N_{T1}^2(k_{x_1} \mid x_1)$	2	0	2	2	2	$N_{T1}^2(X_1) = 8$

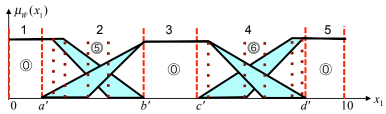
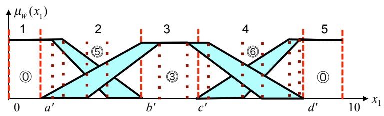
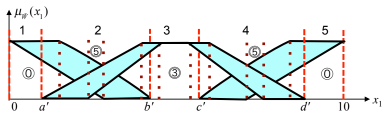
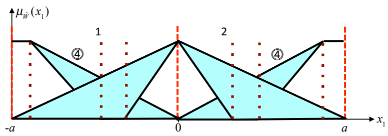
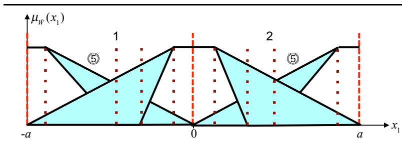
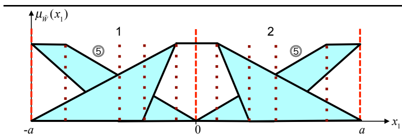
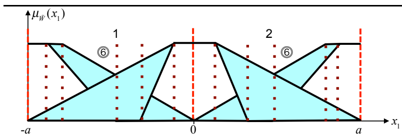


k_{x_1}	1	$2 = N_{T1}^1(X_1)$				
$N_R(k_{x_1})$	2	0	2	2	2	$N_{T1}^2(X_1) = 8$
$N_{T1}^2(k_{x_1} \mid x_1)$	3	3	$N_{T1}^2(X_1) = 6$			



k_{x_1}	1	$2 = N_{T1}^1(X_1)\,2$	
$N_R(k_{x_1})$	2	2	
$N_{T1}^2(k_{x_1} \mid x_1)$	2	2	$N_{T1}^2(X_1) = 4$

Example 3: More examples of IT2 first- and second-order rule partitions. These examples illustrate what happens to the number of first- and second-order rule partitions as some or all of the FOUs change shape.

	k_{x_1}	1	2	3	4	$5 = N^1_{IT2}(X_1)$	
	$N_R(k_{x_1})$	1	2	1	2	1	
	$N^2_{IT2}(k_{x_1} \mid x_1)$	0	5	0	6	0	$N^2_{IT2}(X_1) = 11$
	k_{x_1}	1	2	3	4	$5 = N^1_{IT2}(X_1)$	
	$N_R(k_{x_1})$	1	2	1	2	1	
	$N^2_{IT2}(k_{x_1} \mid x_1)$	0	5	3	6	0	$N^2_{IT2}(X_1) = 14$
	k_{x_1}	1	2	3	4	$5 = N^1_{IT2}(X_1)$	
	$N_R(k_{x_1})$	1	2	1	2	1	
	$N^2_{IT2}(k_{x_1} \mid x_1)$	0	5	3	5	0	$N^2_{IT2}(X_1) = 13$
	k_{x_1}	1	$2 = N^1_{IT2}(X_1)$				
	$N_R(k_{x_1})$	2	2				
	$N^2_{IT2}(k_{x_1} \mid x_1)$	4	4	$N^2_{IT2}(X_1) = 8$			
	k_{x_1}	1	$2 = N^1_{IT2}(X_1)$				
	$N_R(k_{x_1})$	2	2				
	$N^2_{IT2}(k_{x_1} \mid x_1)$	5	5	$N^2_{IT2}(X_1) = 10$			
	k_{x_1}	1	$2 = N^1_{IT2}(X_1)$				
	$N_R(k_{x_1})$	2	2				
	$N^2_{IT2}(k_{x_1} \mid x_1)$	5	5	$N^2_{IT2}(X_1) = 10$			
	k_{x_1}	1	$2 = N^1_{IT2}(X_1)$				
	$N_R(k_{x_1})$	2	2				
	$N^2_{IT2}(k_{x_1} \mid x_1)$	6	6	$N^2_{IT2}(X_1) = 12$			

II. Data for IT2 FPID Controller Novelty Partitions

Table SM-1: Ranges, Rules, UMFs, LMFs and firing intervals in the four regions (see Fig. 8)

	Region 1	Region 2	Region 3	Region 4
Ranges	$E \in [-1,0]$ $\Delta E \in [-1,0]$	$E \in [-1,0]$ $\Delta E \in [0,1]$	$E \in [0,1]$ $\Delta E \in [0,1]$	$E \in [0,1]$ $\Delta E \in [-1,0]$
Rules	$\tilde{R}_Z^1 : (\tilde{N}_E, \tilde{N}_{\Delta E}) \to -1$ $\tilde{R}_Z^2 : (\tilde{N}_E, \tilde{Z}_{\Delta E}) \to -0.5$ $\tilde{R}_Z^4 : (\tilde{Z}_E, \tilde{N}_{\Delta E}) \to -0.5$ $\tilde{R}_Z^5 : (\tilde{Z}_E, \tilde{Z}_{\Delta E}) \to 0$	$\tilde{R}_Z^2 : (\tilde{N}_E, \tilde{Z}_{\Delta E}) \to -0.5$ $\tilde{R}_Z^3 : (\tilde{N}_E, \tilde{P}_{\Delta E}) \to 0$ $\tilde{R}_Z^5 : (\tilde{Z}_E, \tilde{Z}_{\Delta E}) \to 0$ $\tilde{R}_Z^6 : (\tilde{Z}_E, \tilde{P}_{\Delta E}) \to 0.5$	$\tilde{R}_Z^5 : (\tilde{Z}_E, \tilde{Z}_{\Delta E}) \to 0$ $\tilde{R}_Z^6 : (\tilde{Z}_E, \tilde{P}_{\Delta E}) \to 0.5$ $\tilde{R}_Z^8 : (\tilde{P}_E, \tilde{Z}_{\Delta E}) \to 0.5$ $\tilde{R}_Z^9 : (\tilde{P}_E, \tilde{P}_{\Delta E}) \to 1$	$\tilde{R}_Z^4 : (\tilde{Z}_E, \tilde{N}_{\Delta E}) \to -0.5$ $\tilde{R}_Z^5 : (\tilde{Z}_E, \tilde{Z}_{\Delta E}) \to 0$ $\tilde{R}_Z^7 : (\tilde{P}_E, \tilde{N}_{\Delta E}) \to 0$ $\tilde{R}_Z^8 : (\tilde{P}_E, \tilde{Z}_{\Delta E}) \to 0.5$
UMFs	$UMF(\tilde{N}_E) = -E$ $UMF(\tilde{Z}_E) = E+1$ $UMF(\tilde{N}_{\Delta E}) = -\Delta E$ $UMF(\tilde{Z}_{\Delta E}) = \Delta E + 1$	$UMF(\tilde{N}_E) = -E$ $UMF(\tilde{Z}_E) = E+1$ $UMF(\tilde{Z}_{\Delta E}) = 1-\Delta E$ $UMF(\tilde{P}_{\Delta E}) = \Delta E$	$UMF(\tilde{Z}_E) = 1-E$ $UMF(\tilde{P}_E) = E$ $UMF(\tilde{Z}_{\Delta E}) = 1-\Delta E$ $UMF(\tilde{P}_{\Delta E}) = \Delta E$	$UMF(\tilde{Z}_E) = 1-E$ $UMF(\tilde{P}_E) = E$ $UMF(\tilde{N}_{\Delta E}) = -\Delta E$ $UMF(\tilde{Z}_{\Delta E}) = 1+\Delta E$
LMFs	$LMF(\tilde{N}_E) = -0.2E$ $LMF(\tilde{Z}_E) = 0.9(E+1)$ $LMF(\tilde{N}_{\Delta E}) = -0.3\Delta E$ $LMF(\tilde{Z}_{\Delta E}) = 0.9(\Delta E+1)$	$LMF(\tilde{N}_E) = -0.2E$ $LMF(\tilde{Z}_E) = 0.9(E+1)$ $LMF(\tilde{Z}_{\Delta E}) = 0.9(1-\Delta E)$ $LMF(\tilde{P}_{\Delta E}) = 0.3\Delta E$	$LMF(\tilde{Z}_E) = 0.9(1-E)$ $LMF(\tilde{P}_E) = 0.2E$ $LMF(\tilde{Z}_{\Delta E}) = 0.9(1-\Delta E)$ $LMF(\tilde{P}_{\Delta E}) = 0.3\Delta E$	$LMF(\tilde{Z}_E) = 0.9(1-E)$ $LMF(\tilde{P}_E) = 0.2E$ $LMF(\tilde{N}_{\Delta E}) = -0.3\Delta E$ $LMF(\tilde{Z}_{\Delta E}) = 0.9(1+\Delta E)$
Firing intervals	$\tilde{R}_Z^1 \begin{cases} \overline{f}^1 = E \cdot \Delta E \\ \underline{f}^1 = 0.06E \cdot \Delta E \end{cases}$ $\tilde{R}_Z^2 \begin{cases} \overline{f}^2 = -E \cdot (1+\Delta E) \\ \underline{f}^2 = -0.18E \cdot (1+\Delta E) \end{cases}$ $\tilde{R}_Z^4 \begin{cases} \overline{f}^4 = -(1+E) \cdot \Delta E \\ \underline{f}^4 = -0.27(1+E) \cdot \Delta E \end{cases}$ $\tilde{R}_Z^5 \begin{cases} \overline{f}^5 = (1+E) \cdot (1+\Delta E) \\ \underline{f}^5 = 0.81(1+E) \cdot (1+ \end{cases}$	$\tilde{R}_Z^2 \begin{cases} \overline{f}^2 = -E \cdot (1-\Delta E) \\ \underline{f}^2 = -0.18E \cdot (1-\Delta E) \end{cases}$ $\tilde{R}_Z^3 \begin{cases} \overline{f}^3 = -E \cdot \Delta E \\ \underline{f}^3 = -0.06E \cdot \Delta E \end{cases}$ $\tilde{R}_Z^5 \begin{cases} \overline{f}^5 = (1+E) \cdot (1-\Delta E) \\ \underline{f}^5 = 0.81(1+E) \cdot (1-\Delta E) \end{cases}$ $\tilde{R}_Z^6 \begin{cases} \overline{f}^6 = (1+E) \cdot \Delta E \\ \underline{f}^6 = 0.27(1+E) \cdot \Delta E \end{cases}$	$\tilde{R}_Z^5 \begin{cases} \overline{f}^5 = (1-E) \cdot (1-\Delta E) \\ \underline{f}^5 = 0.81(1-E) \cdot (1-\Delta E) \end{cases}$ $\tilde{R}_Z^6 \begin{cases} \overline{f}^6 = (1-E) \cdot \Delta E \\ \underline{f}^6 = 0.27(1-E) \cdot \Delta E \end{cases}$ $\tilde{R}_Z^8 \begin{cases} \overline{f}^8 = E \cdot (1-\Delta E) \\ \underline{f}^8 = 0.18E \cdot (1-\Delta E) \end{cases}$ $\tilde{R}_Z^9 \begin{cases} \overline{f}^9 = E \cdot \Delta E \\ \underline{f}^9 = 0.06E \cdot \Delta E \end{cases}$	$\tilde{R}_Z^4 \begin{cases} \overline{f}^4 = -(1-E) \cdot \Delta E \\ \underline{f}^4 = -0.27(1-E) \cdot \Delta E \end{cases}$ $\tilde{R}_Z^5 \begin{cases} \overline{f}^5 = (1-E) \cdot (1+\Delta E) \\ \underline{f}^5 = 0.81(1-E) \cdot (1+\Delta E) \end{cases}$ $\tilde{R}_Z^7 \begin{cases} \overline{f}^7 = -E \cdot \Delta E \\ \underline{f}^7 = -0.06E \cdot \Delta E \end{cases}$ $\tilde{R}_Z^8 \begin{cases} \overline{f}^8 = E \cdot (1+\Delta E) \\ \underline{f}^8 = 0.18E \cdot (1+\Delta E) \end{cases}$

- Ranges can be deduced from Fig. 8.
- Rules can be created from Table VII.
- UMFs and LMFs can be deduced from Fig. 8.
- Firing intervals are for each region's rules, and use the second line of (1) when: $p = 2$, E is $\tilde{F}_1^l$ and ΔE is $\tilde{F}_2^l$; they also use the product t-norm.

Table SM-2: y_l^{COS} and y_r^{COS} iterations in the four regions. If $y_l^{(j)}$ $(j \in \{0,1,...,4\})$ is the winner, then $L = j$, and if $y_r^{(k)}$ $(k \in \{0,1,...,4\})$ is the winner, then $R = k$ (see Fig. 9)

Region	y_l^{COS} **iterations**	y_r^{COS} **iterations**
1	$y_l^{(0)} = \frac{-\underline{f}^1 - 0.5\underline{f}^2 - 0.5\underline{f}^4}{\underline{f}^1 + \underline{f}^2 + \underline{f}^4 + \underline{f}^5}$ $y_l^{(1)} = \frac{-\overline{f}^1 - 0.5\underline{f}^2 - 0.5\underline{f}^4}{\overline{f}^1 + \underline{f}^2 + \underline{f}^4 + \underline{f}^5}$ $y_l^{(2)} = \frac{-\overline{f}^1 - 0.5\overline{f}^2 - 0.5\underline{f}^4}{\overline{f}^1 + \overline{f}^2 + \underline{f}^4 + \underline{f}^5}$ $y_l^{(3)} = \frac{-\overline{f}^1 - 0.5\overline{f}^2 - 0.5\overline{f}^4}{\overline{f}^1 + \overline{f}^2 + \overline{f}^4 + \underline{f}^5}$ $y_l^{(4)} = \frac{-\overline{f}^1 - 0.5\overline{f}^2 - 0.5\overline{f}^4}{\overline{f}^1 + \overline{f}^2 + \overline{f}^4 + \overline{f}^5}$	$y_r^{(0)} = \frac{-\overline{f}^1 - 0.5\overline{f}^2 - 0.5\overline{f}^4}{\overline{f}^1 + \overline{f}^2 + \overline{f}^4 + \overline{f}^5}$ $y_r^{(1)} = \frac{-\underline{f}^1 - 0.5\overline{f}^2 - 0.5\overline{f}^4}{\underline{f}^1 + \overline{f}^2 + \overline{f}^4 + \overline{f}^5}$ $y_r^{(2)} = \frac{-\underline{f}^1 - 0.5\underline{f}^2 - 0.5\overline{f}^4}{\underline{f}^1 + \underline{f}^2 + \overline{f}^4 + \overline{f}^5}$ $y_r^{(3)} = \frac{-\underline{f}^1 - 0.5\underline{f}^2 - 0.5\underline{f}^4}{\underline{f}^1 + \underline{f}^2 + \underline{f}^4 + \overline{f}^5}$ $y_r^{(4)} = \frac{-\underline{f}^1 - 0.5\underline{f}^2 - 0.5\underline{f}^4}{\underline{f}^1 + \underline{f}^2 + \underline{f}^4 + \underline{f}^5}$
2	$y_l^{(0)} = \frac{-0.5\underline{f}^2 + 0.5\underline{f}^6}{\underline{f}^2 + \underline{f}^3 + \underline{f}^5 + \underline{f}^6}$ $y_l^{(1)} = \frac{-0.5\overline{f}^2 + 0.5\underline{f}^6}{\overline{f}^2 + \underline{f}^3 + \underline{f}^5 + \underline{f}^6}$ $y_l^{(2)} = \frac{-0.5\overline{f}^2 + 0.5\underline{f}^6}{\overline{f}^2 + \overline{f}^3 + \underline{f}^5 + \underline{f}^6}$ $y_l^{(3)} = \frac{-0.5\overline{f}^2 + 0.5\underline{f}^6}{\overline{f}^2 + \overline{f}^3 + \overline{f}^5 + \underline{f}^6}$ $y_l^{(4)} = \frac{-0.5\overline{f}^2 + 0.5\overline{f}^6}{\overline{f}^2 + \overline{f}^3 + \overline{f}^5 + \overline{f}^6}$	$y_r^{(0)} = \frac{-0.5\overline{f}^2 + 0.5\overline{f}^6}{\overline{f}^2 + \overline{f}^3 + \overline{f}^5 + \overline{f}^6}$ $y_r^{(1)} = \frac{-0.5\underline{f}^2 + 0.5\overline{f}^6}{\underline{f}^2 + \overline{f}^3 + \overline{f}^5 + \overline{f}^6}$ $y_r^{(2)} = \frac{-0.5\underline{f}^2 + 0.5\overline{f}^6}{\underline{f}^2 + \underline{f}^3 + \overline{f}^5 + \overline{f}^6}$ $y_r^{(3)} = \frac{-0.5\underline{f}^2 + 0.5\overline{f}^6}{\underline{f}^2 + \underline{f}^3 + \underline{f}^5 + \overline{f}^6}$ $y_r^{(4)} = \frac{-0.5\underline{f}^2 + 0.5\underline{f}^6}{\underline{f}^2 + \underline{f}^3 + \underline{f}^5 + \underline{f}^6}$
3	$y_l^{(0)} = \frac{0.5\underline{f}^6 + 0.5\underline{f}^8 + \underline{f}^9}{\underline{f}^5 + \underline{f}^6 + \underline{f}^8 + \underline{f}^9}$ $y_l^{(1)} = \frac{0.5\underline{f}^6 + 0.5\underline{f}^8 + \underline{f}^9}{\overline{f}^5 + \underline{f}^6 + \underline{f}^8 + \underline{f}^9}$ $y_l^{(2)} = \frac{0.5\overline{f}^6 + 0.5\underline{f}^8 + \underline{f}^9}{\overline{f}^5 + \overline{f}^6 + \underline{f}^8 + \underline{f}^9}$ $y_l^{(3)} = \frac{0.5\overline{f}^6 + 0.5\overline{f}^8 + \underline{f}^9}{\overline{f}^5 + \overline{f}^6 + \overline{f}^8 + \underline{f}^9}$ $y_l^{(4)} = \frac{0.5\overline{f}^6 + 0.5\overline{f}^8 + \overline{f}^9}{\overline{f}^5 + \overline{f}^6 + \overline{f}^8 + \overline{f}^9}$	$y_r^{(0)} = \frac{0.5\overline{f}^6 + 0.5\overline{f}^8 + \overline{f}^9}{\overline{f}^5 + \overline{f}^6 + \overline{f}^8 + \overline{f}^9}$ $y_r^{(1)} = \frac{0.5\overline{f}^6 + 0.5\overline{f}^8 + \overline{f}^9}{\underline{f}^5 + \overline{f}^6 + \overline{f}^8 + \overline{f}^9}$ $y_r^{(2)} = \frac{0.5\underline{f}^6 + 0.5\overline{f}^8 + \overline{f}^9}{\underline{f}^5 + \underline{f}^6 + \overline{f}^8 + \overline{f}^9}$ $y_r^{(3)} = \frac{0.5\underline{f}^6 + 0.5\underline{f}^8 + \overline{f}^9}{\underline{f}^5 + \underline{f}^6 + \underline{f}^8 + \overline{f}^9}$ $y_r^{(4)} = \frac{0.5\underline{f}^6 + 0.5\underline{f}^8 + \underline{f}^9}{\underline{f}^5 + \underline{f}^6 + \underline{f}^8 + \underline{f}^9}$
4	$y_l^{(0)} = \frac{-0.5\underline{f}^4 + 0.5\underline{f}^8}{\underline{f}^4 + \underline{f}^5 + \underline{f}^7 + \underline{f}^8}$ $y_l^{(1)} = \frac{-0.5\overline{f}^4 + 0.5\underline{f}^8}{\overline{f}^4 + \underline{f}^5 + \underline{f}^7 + \underline{f}^8}$ $y_l^{(2)} = \frac{-0.5\overline{f}^4 + 0.5\underline{f}^8}{\overline{f}^4 + \overline{f}^5 + \underline{f}^7 + \underline{f}^8}$ $y_l^{(3)} = \frac{-0.5\overline{f}^4 + 0.5\underline{f}^8}{\overline{f}^4 + \overline{f}^5 + \overline{f}^7 + \underline{f}^8}$ $y_l^{(4)} = \frac{-0.5\overline{f}^4 + 0.5\overline{f}^8}{\overline{f}^4 + \overline{f}^5 + \overline{f}^7 + \overline{f}^8}$	$y_r^{(0)} = \frac{-0.5\overline{f}^4 + 0.5\overline{f}^8}{\overline{f}^4 + \overline{f}^5 + \overline{f}^7 + \overline{f}^8}$ $y_r^{(1)} = \frac{-0.5\underline{f}^4 + 0.5\overline{f}^8}{\underline{f}^4 + \overline{f}^5 + \overline{f}^7 + \overline{f}^8}$ $y_r^{(2)} = \frac{-0.5\underline{f}^4 + 0.5\overline{f}^8}{\underline{f}^4 + \underline{f}^5 + \overline{f}^7 + \overline{f}^8}$ $y^{(3)} = \frac{-0.5\underline{f}^4 + 0.5\overline{f}^8}{\underline{f}^4 + \underline{f}^5 + \underline{f}^7 + \overline{f}^8}$ $y_r^{(4)} = \frac{-0.5\underline{f}^4 + 0.5\underline{f}^8}{\underline{f}^4 + \underline{f}^5 + \underline{f}^7 + \underline{f}^8}$

To obtain the formulas which are in this table:

- $y_l^{(j)}$: Use (2), the numbering of the fired rules in each region (from Table SM-1), and the crisp rule consequents (see "Rules" in Table SM-1), which play the role of the $c_l(\tilde{G}^i)$ in (2).
- $y_r^{(k)}$: Use (3), the numbering of the fired rules in each region (from Table SM-1), and the crisp rule consequents (see "Rules" in Table SM-1), which play the role of the $c_r(\tilde{G}^i)$ in (3). Note that because the rule consequents are crisp, $c_r(\tilde{G}^i) = c_l(\tilde{G}^i)$.

III. Control Surfaces for IT2 FPID Controllers Designs

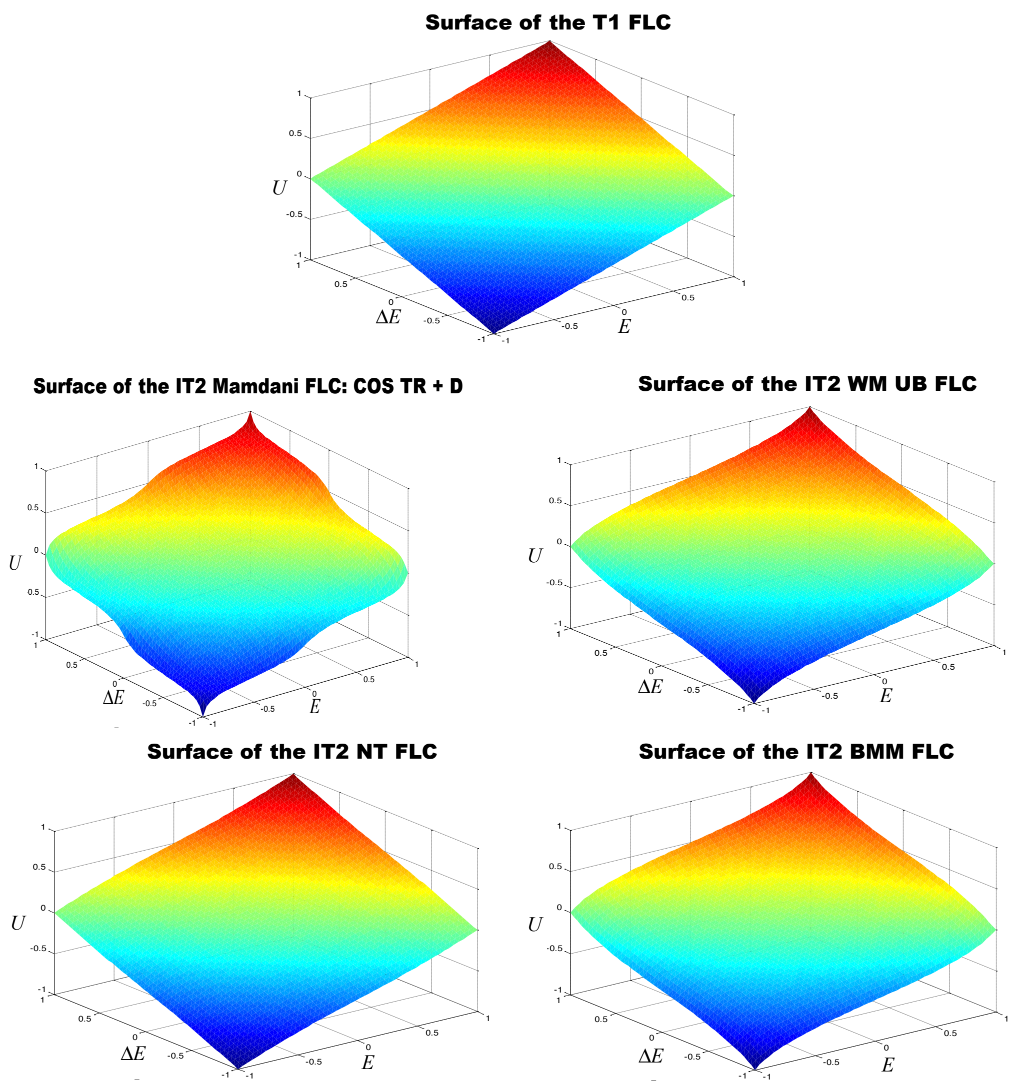


Figure SM-1: Control surfaces for T1 FLC and the four IT2 FLCs (Mendel, 2017 [37, p. 606]; © 2017, Springer). "D" denotes defuzzification. The figures were generated, most graciously, by Prof. Tufan Kumbasar.

Control surfaces for five FPID controllers are given in Fig. SM-1. The following is stated in [37, p. 695]: "Comparing the T1, COS TR + D, WM UB, NT and BMM control surfaces in this figure, a number of observations can be made:

1. "The four adjacent end-points of the surfaces are the same, and it is only the ways in which the adjacent end-points are connected that are different.
2. "The adjacent end-points of the T1 FLC surface are connected by straight lines.
3. "The adjacent end-points of each of the four IT2 FLC surfaces are connected by curves, each one of which provides a different kind of interpolation between the adjacent end-points, and so it is in this sense that one may say that the IT2 FLC surfaces are smoother than the T1 FLC surface.

4. “The IT2 Mamdani FLC: COS TR + D curves exhibit the most varying curvature, followed by either the IT2 WM UB FLC or the IT2 BMM FLC, whereas the IT2 NT FLC curves are the least varying.
5. “The IT2 WM UB, NT and BMM adjacent end-point curves can also be interpreted as different ways to smooth out the more varying curvature of the IT2 Mamdani FLC: COS TR + D adjacent end-point curves.”